$\mathcal{H}_2$ OPTIMAL SENSING ARCHITECTURE WITH MODEL UNCERTAINTY

A Thesis

by

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ABSTRACT

In this thesis, I shall present an integrated approach to control and sensing design. The framework assumes sensor noise as a design variable along with the controller and determines l_1 regularized optimal sensing precision. This design satisfies a given closed-loop performance in the presence of model uncertainty. Two methods will be proposed to achieve this.

The first method designs a controller for an open loop uncertain system, which is scaled in order to have a finite $\mathcal{H}_2$ norm. Within this, two approaches have been pursued. In the first approach, uncertainty has been represented as polytopic and, in the second formulation, modelled using integral quadratic constraints (IQC). These two approaches have been applied to an active suspension control and sensing design problem and demonstrate that the IQC based approach provides better results and is able to incorporate larger system uncertainty.

The second method finds an appropriate scaling to bound the $\mathcal{H}_2$ norm of an uncertain controlled system. The sensor precision is found as the minimal solution to an optimization problem. The design is tested for stability and robustness on a tensegrity robot arm model.

DEDICATION

To my father, without whom I would never have ventured into a foreign land;

to my grandfather, who supported me;

and to Aggieland.

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All other work conducted for the thesis was completed by the student independently.

NOMENCLATURE

BMI	Bi-Linear Matrix Inequality
IQC	Integral Quadratic Constraint
LFT	Linear Fractional Transformation
LMI	Linear Matrix Inequality
LQG	Linear Quadratic Gaussian
LTI	Linear Time Invariant
RMS	Root Mean Square

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1. INTRODUCTION AND PROBLEM STATEMENT

1.1 Introduction

Control system design entails choosing an optimal control law to achieve a particular goal. Any system can be divided into two parts, namely the control architecture, which consists of the controller and its communication links, and the information architecture, which is the real time data gathered from the network of sensors and actuators. The output performance of the system depends on both of these parts. Traditionally, information architecture is fixed and the controller is designed to optimize closed loop performance with the least possible control effort. The performance of the system is limited by the accuracy and location of the of the available sensors and actuators.

The conventional approach has several shortcomings. The limits of performance due to the system architecture may prevent the designer from achieving the required performance. More components mean more noise and larger errors; there might be issues such as cross-talk or over-correction. An *ad-hoc* method of placing these components does not allow for flexibility in design of the architecture. Even if architecture changes were allowed, it is not clear which components to keep or eliminate that will contribute to the desired performance. Many of the components may be unnecessary or may have more than required precision. This results in poor use of system resources and yields an unnecessarily expensive design. Thus, the predetermined system architecture is an impediment in the optimal control system design.

In order to achieve a truly optimal design, integration of the design of information architecture with control and estimation design is needed. A system-level optimization problem allows the sensor and actuator precisions to be design variables along with the controller. This approach determines the optimal location and precision of the components to achieve a desired closed loop performance. Through this work, a co-design of the sensing architecture and the controller is proposed. The actuator architecture is assumed to be predetermined.

Furthermore, the main contribution of this work is applying these principles to an uncertain system. That is, we are interested in determining locations and precisions of sensors, from a dictionary of possible sensors, that will achieve a desired $\mathcal{H}_2$ closed-loop performance, in the presence of model uncertainty. In this work, integral quadratic constraints (IQCs) are used to describe the performance of the uncertain system.

1.1.1 Prior art

The information architecture problem has been first addressed by Li *et al.* [1] in 2008 where the actuator and the sensing architecture, including precision of components, was integrated with $\mathcal{H}_2$ optimal dynamic output-feedback control design. More recently, the problem of optimal architecture design has been considered in the context of large-scale system design where the choices for sensor and actuator location and precision are not straightforward. Schuler *et al.* [2] considers design of output feedback control system with fixed structure for discrete-time interconnected systems. The number of measurement links are minimized using successive weighted l_1 optimization problems. Lin *et al.* [3] use similar l_1 optimization framework to design sparse and block sparse feedback gains in an $\mathcal{H}_2$ optimization framework. Very recently, the work of Matni *et al.* [4] present a framework called regularization for design that generalizes the work described above and in the reference within [4]. The focus in that work is on determining sparse communication and actuation topology and optimizing system-level $\mathcal{H}_2$ optimal performance using full-state feedback control. In all the above described work, the sensor and actuator noise characteristics have been assumed to be known. To the best of our knowledge, the work of Li *et al.* [1] is the only work that determines the requires sensor and actuator noise in the system-level optimization.

$\mathcal{H}_2$ optimal design is solved for uncertain systems in [5], [6] and [7]. The work of Iwasaki *et al.* [5] proves that the robust $\mathcal{H}_2$ problem for uncertain systems can be solved by the Finsler's lemma and gives a formula for the controller to achieve this. Megretski *et al.* [7] elaborate on using IQCs to characterize properties of uncertain signals, and exploit the structure of these systems to find upper bounds on quadratic costs associated with them. The nature of uncertainties is studied in great detail in [6], where the concepts of robust well-connectedness and robust stability are introduced and used to design controllers for various types of uncertainties. The conservativeness of the bounds for $\mathcal{H}_2$ optimal design given by IQCs are explored. These works find stabilizing controllers for systems with model uncertainty, assuming a fixed information architecture.

1.1.2 Contribution of this work

This research extends [1] by including model uncertainty in the design optimization and combining l_1 regularization to determine both the required precision and location of the sensors, and the dynamic output-feedback controller, which achieves the desired $\mathcal{H}_2$ performance. The uncertainty is modelled in the system using IQCs (integral quadratic constraints [7] and compared to that modelled as polytopic uncertainty.

1.1.3 Structure of the work

Two different approaches are explored while marrying the concepts of information architecture and model uncertainty. Section 2 motivates information architecture and $\mathcal{H}_2$ robust control for tensegrity systems. Section 3 describes the first method of designing the sensing architecture of a stable system but with uncertain parameters. Further, it discusses the possibility of solving for a stabilizing controller while simultaneously optimizing for sensing precision in uncertain unstable systems. Section 4 offers results and conclusions based on applying these techniques to the models using MATLAB.

The optimization problems in this work are all based on linear matrix inequalities (LMI). We employ the LMI solver *'cvx'* [8] and MATLAB [9] Robust Control toolbox to define linear fractional transformations (LFT).

1.2 Problem statement

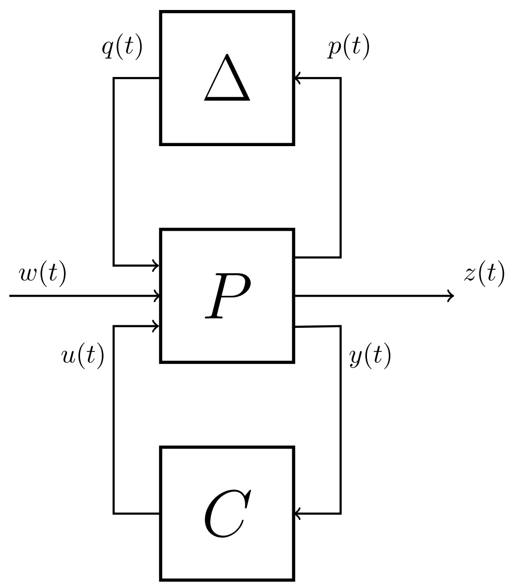

Figure 1.1: Representation of uncertain control system

Consider a linear time-invariant (LTI) system P,

$$\dot{x} = Ax + B_u u + D_d w_d + D_a w_a, \tag{1.1}$$

$$z = C_z x, \tag{1.2}$$

$$y = C_y x + D_{yw} w_s, \tag{1.3}$$

which is also the nominal system without uncertainty ($\Delta = 0$), where $x \in \mathbb{R}^{n_x}$ is the state vector of the system, $z \in \mathbb{R}^{n_z}$ is the controlled output vector, $y \in \mathbb{R}^{n_y}$ is the measurement vector. The exogenous signals are denoted by, $w_d \in \mathbb{R}^{n_d}$ process noise, $w_s \in \mathbb{R}^{n_y}$ sensor noise and $w_a \in \mathbb{R}^{n_u}$ actuator noise.

The controller C is a dynamic output feedback controller defined as,

$$\dot{x}_c = A_c x_c + B_c y, \tag{1.4}$$

$$u = C_c x_c + D_c y, \tag{1.5}$$

By introducing model uncertainty Δ, the dynamic system can be expressed as,

$$\dot{x} = A(\Delta)x + B_u(\Delta)u + D_d(\Delta)w_d + D_a(\Delta)w_a, \tag{1.6}$$

$$z = C_z(\Delta)x, \tag{1.7}$$

$$y = C_y(\Delta)x + D_{yw}(\Delta)w_s, \tag{1.8}$$

or alternatively as,

$$\begin{bmatrix} \dot{x} \\ p \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & B_q & B_w & B_u \\ C_p & 0 & 0 & 0 \\ C_z & 0 & 0 & 0 \\ C_y & 0 & D_{yw} & 0 \end{bmatrix} \begin{bmatrix} x \\ q \\ w \\ u \end{bmatrix} \tag{1.9}$$

$$q = \Delta p \tag{1.10}$$

where

$$B_w := \begin{bmatrix} D_d & D_a & 0 \end{bmatrix},$$

and the uncertain signals $q \in \mathbb{R}^{n_q}$ and $p \in \mathbb{R}^{n_p}$.

In this work, we assume the noise intensity of w_d and w_a are known and specified by W_d and W_a. The sensor noise intensity W_s is *unknown* and is determined from the optimization. The system matrices are uncertain, where the uncertainty is represented by Δ.

The objective of this thesis is to design $\mathcal{H}_2$ optimal dynamic output-feedback controller $u := C(s)y$, and determine sensor noise intensity W_s such that the closed-loop performance

$$\mathbf{E}\left[zz^T\right] \leq \bar{Z}, \text{ and } \mathbf{E}\left[uu^T\right] \leq \bar{U}, \tag{1.11}$$

with given $\bar{Z}$ and $\bar{U}$, is satisfied.

In the design formulation, we include a dictionary of all available sensors, and use l_1 regularization on W_s^{-1} to determine the sparse sensing architecture that meets the required $\mathcal{H}_2$ performance. We introduce a new variable associated to precision, $\Gamma_s := W_s^{-1}$ and note that $\Gamma_s = \mathbf{diag}(\gamma_s)$, where $\gamma_s \in \mathbb{R}^{n_y} \geq 0$ is the vector of decision variables associated with Γ_s. We also introduce $\rho_s \in \mathbb{R}^{n_y} > 0$ as the price per unit precision for the sensor array.

The optimization problem for synthesis is therefore

$$\min_{\gamma_s, C(s)} \|\rho_s \circ \gamma_s\|_1, \tag{1.12}$$

subject to (1.6), (1.7), (1.8), (1.11) and $u := C(s)y$, where $\circ$ denotes the Hadamard product.

1.3 Tensegrity model and dynamics

The design framework in this thesis is applied to a tensegrity model and an optimal sensing architecture for such a system is proposed. Let us first understand what tensegrity structures are and look at their dynamics and modelling.

1.3.1 Brief background on Tensegrity

Tensegrity structures are mainly structures comprising bars, constituting the compressible members and strings, which are the tensile members, that are in stable equilibrium. The bars can connected to each other through a ball joint, but not through pin joints, that impart torque. The integrity and flexibility of these structures depends on the tensile members, whereas the bars are hard and axially loaded (in compression). When acted on by a force these the shape of this structure deforms slightly to counter these forces. They are capable of large displacement and are deployable [10].

These structures can be used to deliver payloads or used to construct robotic arms that are programmable manipulators. They can be used to carry or pick up objects of an unknown mass. Thus, this is a type of uncertainty that will present itself in the modelling of these structures. In this work, we assume the structures have a parametric uncertainty where the mass might vary from one to up to twenty percent.

Sensors and actuators can be embedded in both kinds of members and thus, these structures are controllable. Sensors measuring the compression in the bars provide the information and actuators can be employed to alter the tension in the strings. A variety of sensors including accelerometers, gyroscopes, displacement sensors, velocity measurement devices, strain gauges and such can be used to measure various parameters along this structure. We will see in this work, which among these is essential to achieve desired robust $\mathcal{H}_2$ performance in presence of uncertainty.

1.3.2 Dynamic equations governing the Tensegrity model

The model as shown in figure Fig.(1.2) is a stable tensegrity structure. The equations governing this model are derived analytically using the MATLAB symbolic toolbox on the basis of Lagrangian mechanics. The state vector and the controlled output vector of the system are given in (1.13).

$$x(t) = \begin{bmatrix} \Theta_1 \\ \Phi_1 \\ \Theta_2 \\ \Phi_2 \\ \dot{\Theta}_1 \\ \dot{\Phi}_1 \\ \dot{\Theta}_2 \\ \dot{\Phi}_2 \end{bmatrix} \text{ and } z(t) = \begin{bmatrix} x_6 \\ y_6 \end{bmatrix}. \tag{1.13}$$

Other details of the model are as given,

$$n_u = 8, \tag{1.14}$$

$$n_y = 32, \tag{1.15}$$

$$n_z = 2. \tag{1.16}$$

A total of thirty-two sensors are attached to this structure. The two dimensional position and velocity $(x, y, \dot{x}, \dot{y})$ is measured at every point 1 through 6. The angular displacements Θ_1 and Θ_2 are measured, along with the angular velocities $\dot{\Theta}_1$ and $\dot{\Theta}_2$. Additionally, we also measure the angular displacements of the two parts within the structure separately, as Φ_1 and Φ_2 and the angular velocities $\dot{\Phi}_1$ and $\dot{\Phi}_2$. There are eight actuators, which are the eight strings that form the tensile members of this structure. The output is the position of the mass attached to the point 6. The objective is to ensure robust $\mathcal{H}_2$ performance in the presence of uncertainty in the mass m and disturbance w.

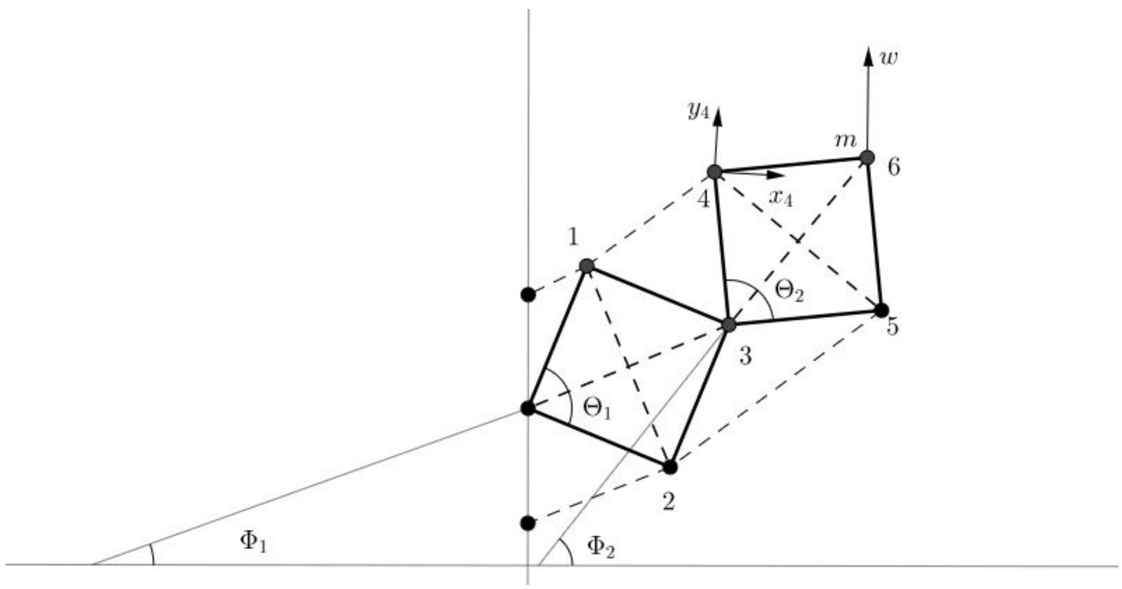

Figure 1.2: Tensegrity Model (1.13)

This section deals with the information architecture problem for certain systems. It discusses control system design for the nominal plant and extends the problem to include $\mathcal{H}_2$ performance criteria. Finally, the control algorithm is applied to design the architecture for a tensegrity model.

2.1 Motivation for covariance as the performance metric

Second order information can be more insightful than first order metrics and many properties of a system can be captured by second order information about the state $x(t)$. Besides covariance, the second order information can be used to quantify robust $\mathcal{H}_2$ performance and disturbance attenuation, spectral properties like power spectral density, and even signal-to-noise ratio of the constituent components.

Covariance of a signal $x(t-\tau)$, such that $x \in \mathbb{R}^{n_x}$ is given as,

$$X := C_{xx}(0) = \sum_{i=1}^{nx} \int_0^{\infty} x_i(t) x_i^T(t) dt = \mathbf{E}\left[x(t)x^T(t)\right], \tag{2.1}$$

where $\tau = 0$, since this is the time-invariant case. Output covariance, especially, is a physically significant measure of the system performance. It relates to the root mean square value of the output, which can be recorded by observing the physical impulse responses of the system. So, for the controlled output $z = C_z x$ from (1.2) at steady state as $t \to \infty$,

$$z_{RMS} := [Z]^{1/2} = \mathbf{E}_{\infty}\left[zz^T\right]^{1/2} = [C_z X C_z^T]^{1/2}. \tag{2.2}$$

The steady state covariance of the exogenous signals w_d, w_a and w_s from (1.1) and (1.3) is represented by,

$$\mathbf{E}_\infty \left[w_i w_i^T\right] = W_i \delta(t-\tau) \text{ for } i = d, a, s. \tag{2.3}$$

Since these signals are independent Gaussian white noises, the noise intensities associated with them, namely W_d, W_a, and W_s are diagonal matrices.

The most promising factor in choosing covariance to quantify performance is that it also sheds light on the stability of the system. Quadratic stability of the system can be established by the Lyapunov equation.

$$0 = AX + XA^T + B_w W B_w^T, \tag{2.4}$$

where X is as defined in (2.1). Note that if (2.4) is satisfied it indicates that the eigen values of A lie in the left half plane and also that the covariance W is positive definite.

2.2 $\mathcal{H}_2$ norm optimization

The $\mathcal{H}_2$ norm of the output z for a system subject to the signal w having a noise intensity W (diagonal) is defined in two ways. The first is the impulse response interpretation, where w is a weighted impulse vector $w(t) = w\delta(t)$, where w_i is a constant weight.

$$\sum_{i=1}^{n_z} ||z_i||_2^2 = ||P\sqrt{W}||_2^2, \tag{2.5}$$

and W is the same as noise intensity, but also can be written as,

$$W = \begin{bmatrix} w_1^2 & 0 & \cdots \\ 0 & w_2^2 & \cdots \\ 0 & \cdots & w_{n_w}^2. \end{bmatrix} \tag{2.6}$$

The second is the stochastic interpretation, and the power associated with w is given $\hat{W}(jw)$. Here,

$$\mathbf{E}\left[|w(t)|^2\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{W}(jw)dw, \tag{2.7}$$

and the output norm is,

$$\begin{aligned} ||z||_2^2 = \mathbf{E}\left[|z(t)|^2\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{Tr}[\hat{P}^*(jw)\hat{W}(jw)\hat{P}(jw)]dw \\ &= W||\hat{P}(jw)||_2^2 \end{aligned} \tag{2.8}$$

The induced norm of the system is calculated as,

$$\begin{aligned} ||\hat{P}(jw)||^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{Tr}[\hat{P}^*(jw)\hat{P}(jw)]dw \\ &= ||P(t)||^2 \\ &= \int_0^{\infty}\mathbf{Tr}[C_z e^{At}B_w B_w^T e^{A^T t}C_z^T]dt \\ &= \mathbf{Tr}\left[C_z\left(\int_0^{\infty}e^{At}B_w B_w^T e^{A^T t}dt\right)C_z^T\right] \\ &= \mathbf{Tr}[C_z X_c C_z^T] && (2.9) \\ &= \mathbf{Tr}[B_w^* Y_o B_w]. && (2.10) \end{aligned}$$

X_c is the controllablity gramian, which by definition is,

$$X_c = \left(\int_0^{\infty}e^{At}B_w W B_w^T e^{A^T t}dt\right), \tag{2.11}$$

and Y_o is observability gramian, which is,

$$Y_o = \left(\int_0^{\infty}C_z^T e^{A^T}te^{At}C_z dt\right). \tag{2.12}$$

The gramians are the solutions of the Lyapunov equations:

$$AX_c + X_c A^T + B_w B_w^T = 0, \tag{2.13}$$

$$A^T Y_o + Y_o A + C_z^T C_z = 0. \tag{2.14}$$

If for some constant $\eta > 0$ and for any positive definite variable $X - X_c <= 0$ that satisfies the Lyapunov inequality,

$$AX + XA^T + B_w W B_w^T < 0, \tag{2.15}$$

$$\mathbf{Tr}[C_z X C_z^T] < \eta, \tag{2.16}$$

and if for any positive definite variable $Y - Y_o <= 0$ that satisfies the Lyapunov inequality,

$$A^T Y + YA + C_z^T C_z < 0, \tag{2.17}$$

$$\mathbf{Tr}[W B_w^T Y B_w] < \eta, \tag{2.18}$$

then the system output norm from (2.8),

$$\sum_{i=1}^{n_z} ||z_i||_2^2 = \mathbf{Tr}[C_z X C_z^T] \tag{2.19}$$

$$= \mathbf{Tr}[W B_w^T Y B_w] \tag{2.20}$$

$$< \eta \tag{2.21}$$

2.2.1 Energy to peak gain of the system

The $\mathcal{H}_2$ performance is expressed as impulse to energy gain in (2.5), similarly there is another system gain related to the $\mathcal{H}_2$ norm of the system,

$$T_{ep} := \sup_{||w||_{\mathcal{L}_2}} ||z||_{\mathcal{L}_\infty}. \tag{2.22}$$

2.3 Information architecture as an $\mathcal{H}_2$ problem

Integrating information architecture with control and estimation design ensures an optimal system. The approach looks to obtain a fair trade off between manufacturing tolerance, the signal processing constraints, the sensor accuracy and the control law. The information architecture problem described in [1] imposes the following constraints on the system defined by (1.1), (1.2) and (1.3);

$$\mathbf{E}\left[zz^T\right] \leq \bar{Z}, \tag{2.23}$$

$$\mathbf{E}\left[uu^T\right] \leq \bar{U}, \tag{2.24}$$

$$\gamma_s \leq \bar{\gamma}_s, \tag{2.25}$$

$$\gamma_a \leq \bar{\gamma}_a, \tag{2.26}$$

$$\$ \leq \bar{\$}, \tag{2.27}$$

where z, u and γ_s are the controlled output, control input and sensor precision decision variable respectively. In [1], the actuator architecture is also unknown, and is thus associated with the decision variable γ_a, where $\gamma_s \in \mathbb{R}^{n_u} \geq 0$.

The cost associated with the sensing architecture, parametrized by γ_s, is given by $\rho_s^T \gamma_s$. Since $\rho_s > 0$ and $\gamma_s \geq 0$, the cost function is equivalent to the weighted l_1 norm $\|\rho_s \circ \gamma_s\|_1$, where $\circ$ denotes the Hadamard product. Similarly, for the actuator architecture parametrized by γ_a, the price per precision for the actuator array is defined as $\rho_a \in \mathbb{R}^{n_u} > 0$ and the corresponding cost is given by $\rho_a^T \gamma_a$. Li *et al.* introduce a new variable associated to actuator precision, $\Gamma_a := W_a^{-1}$ and note that $\Gamma_a = \textbf{diag}(\gamma_a)$. The precisions are bounded by the real world values $\bar{\gamma_s}$ and $\bar{\gamma_a}$ in (2.25) and (2.26), which are respectively the maximum available precisions in the market for the sensors and actuators.

The constraint (2.27) introduces an economic perspective on the problem. The total price of the instrument is given by,

$$\$ = \rho_s^T \gamma_s + \rho_a^T \gamma_a. \tag{2.28}$$

A budget $\bar{\$}$ is forced on the total price of the instrument to make sure that the system is economically feasible.

The closed loop performance is measured in terms of the output covariance (2.23) and the control input covariance (2.24). To draw a parallel between the information architecture problem and the $\mathcal{H}_2$ norm optimization problem should be straightforward. The following theorem gives an exact relation between the two.

Theorem 1. *The minimum solution of the sensing architecture problem as defined in* (2.23)*,* (2.24)*,* (2.25) *and* (2.27) *is equivalent to the $\mathcal{H}_2$ optimization problem,*

$$\min_{\gamma_s, C(s)} ||\rho_s \circ \gamma_s||_1,$$

subject to

$$||P||_2^2 < \mathbf{Tr}\bar{Z} + \mathbf{Tr}\bar{U}, \tag{2.29}$$

assuming the actuator architecture is fixed.

Proof. Consider the system in (1.1), (1.2) and (1.3), closing the loop with the controller given in (1.4) and (1.5), the closed loop system can be written as,

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A + B_u D_c C_y & B_u C_c \\ B_c C_y & A_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} B_w + B_u D_c D_{yw} \\ B_c D_{yw} \end{bmatrix} w, \tag{2.30}$$

$$\begin{bmatrix} z \\ u \end{bmatrix} = \begin{bmatrix} C_z & 0 \\ D_c C_y & C_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} 0 \\ D_c D_{yw} \end{bmatrix} w. \tag{2.31}$$

Thus, with a strictly proper controller $D_c = 0$,

$$\dot{\bar{x}} = A_{cl}\bar{x} + B_{cl}w, \tag{2.32}$$

$$z = C_{cl}\bar{x}, \tag{2.33}$$

$$u = E_{cl}\bar{x} + F_{cl}w. \tag{2.34}$$

The information architecture problem for the closed loop system is given by following linear matrix inequalities (LMIs),

$$A_{cl}X + XA_{cl}^T + B_{cl}WB_{cl}^T < 0, \tag{2.35}$$

$$C_{cl}XC_{cl}^T < \bar{Z}, \tag{2.36}$$

$$E_{cl}^T X E_{cl} < \bar{U}, \tag{2.37}$$

where

$$W = \begin{bmatrix} W_d & 0_{n_d \times n_u} & 0_{n_d \times n_y} \\ \bullet & W_a & 0_{n_u \times n_y} \\ \bullet & \bullet & 0_{n_y \times n_y} \end{bmatrix}. \tag{2.38}$$

The symbol $\bullet$ represents the symmetric terms.

Now, looking at the $\mathcal{H}_2$ problem in terms of the stochastic interpretation. Note that here, the system has one input w and two outputs z and u. For purposes of this problem, the closed loop norm comprises of the norms from $w \mapsto z$ and $w \mapsto u$.

$$\begin{aligned}
||P||_2^2 &= \mathbf{Tr}\left[\begin{bmatrix} C_{cl} \\ E_{cl} \end{bmatrix}\left(\int_0^\infty e^{A_{cl}t} B_{cl} W B_{cl}^T e^{A_{cl}^T t} dt\right)\begin{bmatrix} C_{cl}^T & E_{cl}^T \end{bmatrix}\right] \\
&= \mathbf{Tr}\left[\begin{bmatrix} C_{cl} \\ E_{cl} \end{bmatrix} X_c \begin{bmatrix} C_{cl}^T & E_{cl}^T \end{bmatrix}\right] \\
&= \mathbf{Tr}[C_{cl} X_c C_{cl}^T] + \mathbf{Tr}[E_{cl} X_c E_{cl}^T] \qquad (2.39) \\
&< \mathbf{Tr}\bar{Z} + \mathbf{Tr}\bar{U} \qquad (2.40)
\end{aligned}$$

where X_c is the controllability Gramian of the system and is also a possible solution of the Lyapunov equation (2.4) and will satisfy the Lyapunov inequalities (2.35), (2.36) and (2.37).

If the constraint (2.27) is minimized when γ_a and ρ_a are fixed in the total price of the instrument (2.28) and ρ_s is given, then the constraint (2.25) is absorbed into it.

$$\min_{\gamma_s} \$ \qquad (2.41)$$

$$= \min_{\gamma_s} \rho_s^T \gamma_s, \qquad (2.42)$$

which is equivalent to,

$$\min_{\gamma_s, C(s)} \|\rho_s \circ \gamma_s\|_1. \qquad (2.43)$$

$\square$

2.3.1 LMIs for the information architecture problem

Theorem 2. *If there exists $\gamma_s \in \mathbb{R}^{n_y} \geq 0$, $X = X^T > 0 \in \mathbb{R}^{n_x \times n_x}$, $Y = Y^T > 0 \in \mathbb{R}^{n_x \times n_x}$, $L \in \mathbb{R}^{n_u \times n_x}$, $F \in \mathbb{R}^{n_x \times n_y}$, $Q \in \mathbb{R}^{n_x \times n_x}$, that solves the optimization problem*

$$\min_{\gamma_s, C(s)} \|\rho_s \circ \gamma_s\|_1, \tag{2.44}$$

subject to

$$\begin{bmatrix} \bar{Z} & C_z X & C_z \\ \bullet & X & I_{n_x \times n_x} \\ \bullet & \bullet & Y \end{bmatrix} > 0, \tag{2.45}$$

$$\begin{bmatrix} \bar{U} & L & 0_{n_u \times n_x} \\ \bullet & X & I_{n_x \times n_x} \\ \bullet & \bullet & Y \end{bmatrix} > 0, \tag{2.46}$$

$$\begin{bmatrix} \Phi_{11} + \Phi_{11}^T & \Phi_{12} \\ \bullet & \Phi_{22} \end{bmatrix} < 0, \tag{2.47}$$

$$\Phi_{11} := \begin{bmatrix} AX + B_u L & A \\ Q & YA + FC_y \end{bmatrix}, \tag{2.48}$$

$$\Phi_{12} := \begin{bmatrix} D_d & D_a & 0_{n_x \times n_y} \\ YD_a & YD_a & FD_{yw} \end{bmatrix}, \tag{2.49}$$

$$\Phi_{22} := \begin{bmatrix} -W_d^{-1} & 0_{n_d \times n_u} & 0_{n_d \times n_y} \\ 0_{n_u \times n_d} & -\Gamma_a & 0_{n_u \times n_y} \\ 0_{n_y \times n_d} & 0_{n_y \times n_u} & -\Gamma_s \end{bmatrix}, \tag{2.50}$$

assuming actuator architecture is fixed and the process and actuator noise densities are known, then there exists as controller such that cost and performance constraints are sat-

isfied:

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} V^{-1} & -V^{-1}YB_u \\ 0 & I \end{bmatrix} X \begin{bmatrix} Q - YAX & F \\ L & 0 \end{bmatrix} \begin{bmatrix} U^{-1} & 0 \\ -C_y X U^{-1} & I \end{bmatrix}, \tag{2.51}$$

where V and U are satisfying $YX + VU + I$.

Proof. Refer to [1] and (2.43) from the proof of Theorem 1. □

2.4 Example

Consider the convex optimization problem (2.44), subject to (2.45), (2.46) and (2.47) and assume that actuator architecture is fixed, it can be solved for the tensegrity model given in (1.13).

Let $\bar{Z} = aI_{n_z \times n_z}$, and $\bar{U} = bI_{n_u \times n_u}$, and we can see the optimization chooses a subset of sensors from a dictionary of 32 sensors. The parameters of the simulation are,

$$Mass = 10\text{kg}, \tag{2.52}$$

$$W_d = 0.01, \tag{2.53}$$

$$Uncertainty = 0\%, \tag{2.54}$$

$$\rho_s = ones(n_y). \tag{2.55}$$

Fig.(2.1) shows the different sets of sensors required to achieve a desired closed loop performance $||G_{w \mapsto z}||$. We can observe that as we demand a smaller $\mathcal{H}_2$ norm, the more number of sensors we need and those too of higher precision.

For a required output performance of $||G_{w \mapsto z}|| < 0.01$, the optimization selects only one sensor (24), which measures the vertical (y_6) velocity at the point 6 as shown in Fig.(1.2). If we eliminate all the other sensors and run the algorithm again, we find that the system is still able to maintain the $\mathcal{H}_2$ performance within limits and the minimum

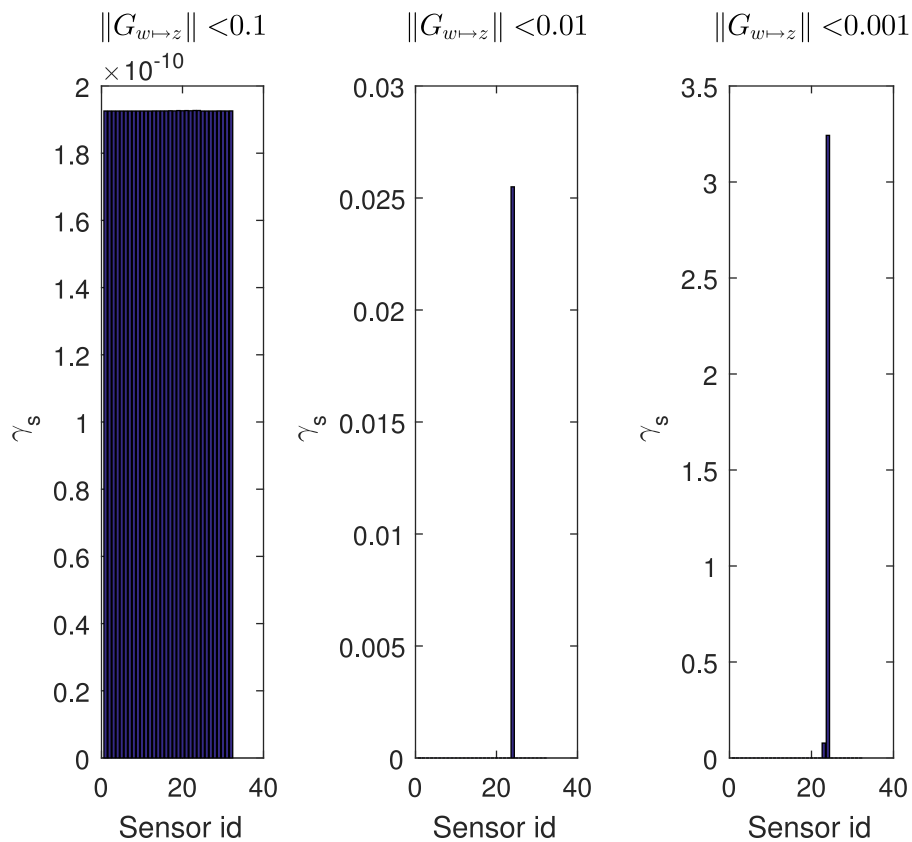

Figure 2.1: Optimal subset of sensors for nominal system (1.13)

precision required to do that is,

$$\gamma_s(24) = 25.49 \times 10^{-3}/m. \tag{2.56}$$

Thus, this shows that the sensing architecture for the nominal system can be as sparse as a single sensor and the system can still achieve the required robust performance.

3. INTEGRATING INFORMATION ARCHITECTURE DESIGN WITH CONTROL AND ESTIMATION FOR UNCERTAIN SYSTEMS

This section begins with an introduction to uncertainty quantification using integral quadratic constraints. Next, the principles of information architecture are derived for an uncertain system. Keep in mind that in this section, we derive the equations using variables W_d and W_a for intensities of process noise and actuator noise, but we assume that both of these are given to us.

3.1 Model uncertainty using IQCs

Model uncertainty stems from imperfect knowledge of the physical system or from the variations in parameter values during operation. Mathematical modeling of systems involves several approximations and assumptions. The actual system may differ greatly from the model, based on which the control system is designed. There is no guarantee that the controller will perform as intended in real world applications. For example, a nonlinear system might be linearized around an equilibrium point in its model and the controller assures performance in these conditions; however during the experiment, the controller might fail when the nonlinear system moves away from the presumed steady state. To overcome these obstacles, the methods of uncertainty quantification are employed. These methods ensure that the control system design is robust in practice. IQCs offer one such method to deal with uncertainties.

3.1.1 Robust well-connectedness and stability

Let us first introduce the notation for system uncertainty in this work. The uncertainty Δ considered here is time varying and norm bounded.

Recall the system equations from (1.9) and (1.10), where z is the controlled output vector, w are the exogenous signals and q and p are the uncertain signals. The transfer function $G_{w \to z}$ can also be expressed as the following,

$$\begin{bmatrix} p \\ z \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} q \\ w \end{bmatrix} \tag{3.1}$$

$$q = \Delta p \tag{3.2}$$

$$w = \begin{bmatrix} w_d \\ w_a \\ w_s \end{bmatrix} \tag{3.3}$$

and written as the upper star product [6] (or upper linear fractional transformation (LFT) [11]) of the uncertainty and the plant.

$$w \mapsto z = P_{22} + P_{21}\Delta \left(I - P_{11}\Delta\right)^{-1} P_{12} := \bar{S}(P, \Delta) \tag{3.4}$$

Robust well-connectedness assures the mapping $w \mapsto z$ is bounded and given by $\bar{S}(P, \Delta)$ for every Δ in a specific subset $\mathbf{\Delta}$ of linear operators on $\mathcal{L}_2$. It depends on the non-singularity of $(I - P_{11}\Delta)$, and in turn on the contractiveness of the operator Δ. This is based on the principles of the small gain principle for robustness of uncertain systems.

We restrict the discussion to arbitrary block structured uncertainties, such that $\Delta \in \Delta_a$ is a contractive operator, where

$$\Delta_a := \left\{ \Delta = \textbf{diag}(\Delta_1, ..., \Delta_d) \in \mathcal{L}_2 \mapsto \mathcal{L}_2,\ ||\Delta||_\infty < 1 \right\}, \tag{3.5}$$

and correspondingly the relation can be expressed as

$$R_a := \left\{ (p, q) \in \mathcal{L}_2 \mapsto \mathcal{L}_2,\ :\ ||E_k p|| \geq ||E_k q|| \right\}, \tag{3.6}$$

for $k = 1, \cdots, n_p$, where the $E_k = [0, ..., 0, I, 0, ..., 0]$. The uncertainty Δ despite being a complex non-linearity can be expressed by a single norm constraint $||E_k p|| \geq ||E_k q||$. In the set of operators Δ_a, the small gain test is conservative, that is, it is not necessary and sufficient. Thus, the condition $||\Delta||_\infty < 1$ is not enough to conclude robust well connectedness. To reduce the conservatism of this bound, we introduce a set of operators that commute with the perturbations. Define a subset of positive definite matrices that commute with $\Delta \in \Delta_a$:

$$\Theta_a = \left\{ \Theta \in \mathcal{L}_2,\ \Theta\Delta = \Delta\Theta,\ \forall\ \Delta \in \Delta_a, \text{such that } \Theta \text{ is invertible} \right\}. \tag{3.7}$$

Note Θ^{-1} also commutes, and the small gain condition changes. Now $I - P\Delta$ is invertible if and only if $I - \Theta P \Theta^{-1}\Delta$ is invertible. If P is time invariant bounded operator on $\mathcal{L}_2[0, \infty)$, robust well connectedness of $\bar{S}(P, \Delta_a)$ is then same as,

$$\inf_{\Theta \in \Theta_a} ||\Theta P \Theta^{-1}|| < 1, \tag{3.8}$$

where the operator $\Theta \in \Theta_a$ is of the form,

$$\Theta = \begin{bmatrix} \theta_1 I & 0 & ... & 0 \\ 0 & ... & ... & 0 \\ 0 & 0 & ... & \theta_d I \end{bmatrix}. \tag{3.9}$$

For $\theta_k > 0$, the set of positive scaling operators form a convex cone of matrices $\mathcal{P}\Theta_a$, such that if $\Theta \in \Theta_a$, then $\Theta^*\Theta \in P\Theta_a$ and if $\Theta \in P\Theta_a$, then $\Theta^{1/2} \in P\Theta_a$.

Stable systems are insensitive to uncertainty in initial conditions, despite perturbations from equilibrium they are guaranteed to return to stable operation. For any $w \in \mathcal{L}_2$ and any initial conditions the input output should be well-defined. If $w = 0$, the maps of the plant and controller should should tend asymptotically to zero. For zero initial conditions, the input output maps should be bounded. The system will be robustly stable if and only if $\Delta \in \Delta^c$, where Δ^c is the set of causal linear operators. Thus, the necessary and sufficient condition for the system to be robustly well-connected and stable be expressed through the following LMI,

$$P^T \Theta P - \Theta < 0 \tag{3.10}$$

For controller co-design, where C internally stabilizes the plant and gives robust performance, find a scaling Θ that infimizes the scaled gain,

$$\inf_{\Theta \in P\Theta_a, C \in \mathcal{C}} ||\Theta^{1/2} \underline{S}(P, C) \Theta^{-1/2}|| < 1. \tag{3.11}$$

If the norm in (3.11) is the $\mathcal{H}_\infty$ norm then the system is said to be Q-stable. For uncertain systems with block-structured uncertainties, Q-stability implies quadratic stability and is equivalent to robust stability. Then we can find a formulation that minimizes the $\mathcal{H}_2$ norm of the uncertain system. There are a few reasons why we prefer $\mathcal{H}_2$ norm instead of $\mathcal{H}_\infty$,

- H_∞ accounts only for the worst possible behaviour.
- $\mathcal{H}_2$ deals with inputs of a typical statistical nature.
- $\mathcal{H}_2$ is well-motivated metric for performance, while H_∞ is used for stability.
- H_∞ bounds H2 operator. It is the least upper bound and this fact is used while solving for the $\mathcal{H}_2$ optimization problem to ensure robust stability.

3.1.2 Closer look at IQCs

Any robustness result can be translated into an integral quadratic constraint. In fact, the quadratic stability of a system can be established by a single IQC, it confines the domain of the arbitrary search for the Lyapunov function in such systems. In the case of uncertain systems, IQC provides not only a way to prove robust stability but also contains more information about the uncertainty than just the norm [7].

IQC, or integral quadratic constraints are used to define bounds on the robust performance of a system. In the IQC framework, the contractiveness constraint $||E_k p|| \geq ||E_k q||$ for $k = 1, \cdots, n_p$ from (3.6) is redefined as,

$$||E_k p||^2 - ||E_k q||^2 = \psi \begin{pmatrix} p \\ q \end{pmatrix} \geq 0, \tag{3.12}$$

Define a set C_ψ^+ where this inequality holds true,

$$C_\psi^+ := \left\{ \begin{pmatrix} p \\ q \end{pmatrix} : \psi \begin{pmatrix} p \\ q \end{pmatrix} \geq 0 \right\}. \tag{3.13}$$

We can also define another set $C_\psi^{-\epsilon}$, where the inequality does not hold,

$$C_\psi^{-\epsilon} = \left\{ \begin{pmatrix} p \\ q \end{pmatrix} : \psi \begin{pmatrix} p \\ q \end{pmatrix} \leq -\epsilon \begin{vmatrix} p \\ q \end{vmatrix}^2 \right\}. \tag{3.14}$$

Let the relation associated with this infeasibility set be $R_P \subset C_\psi^{-\epsilon}$. Then, R_P in a cone where quadratic form ψ is negative and R_a is the cone where it is non-negative cone. Based on convex separation theory [6], two sets that are convex and disjoint, then there always exists a separating hyperplane between them, if and only if the sets are strictly separated. Thus, the two cones are quadratically separated by ψ. The two cones only intersect at zero.

For Ψ, a self-adjoint operator on $\mathcal{L}_2 \times \mathcal{L}_2$, the uncertainty set is said to satisfy integral quadratic constraint defined in (3.12) if $R_{\Delta_a} \subset C_\psi^+$. Robust stability and well-connectedness can now be said to be equivalent to the existence of an operator Ψ,

$$\psi \begin{pmatrix} p \\ q \end{pmatrix} = \left\langle \begin{pmatrix} p \\ q \end{pmatrix}, \Psi \begin{pmatrix} p \\ q \end{pmatrix} \right\rangle, \tag{3.15}$$

and $< \cdot, \cdot >$ is the inner product. Typically $\hat{\Psi}(jw) = \hat{\Psi}^*(jw)$ is a $\hat{L}_\infty$ function written as,

$$\psi(v) = \langle v, \Psi v \rangle = \frac{1}{2\pi} \int_{\infty}^{\infty} \hat{v}(jw)^* \hat{\Psi}(jw) \hat{v}(jw) dw. \tag{3.16}$$

Any integral quadratic constraint can be written in this form and exploited further to define upper limits on functions.

Therefore, the LMI in (3.10) can be rewritten as,

$$\begin{bmatrix} P^T & I^T \end{bmatrix} \Psi \begin{bmatrix} P \\ I \end{bmatrix} < 0, \tag{3.17}$$

where there exists an operator,

$$\Psi = \begin{bmatrix} \Theta & 0 \\ 0 & -\Theta \end{bmatrix}, \tag{3.18}$$

for $\Theta = E_k \theta_k^* \theta_k \in \mathcal{P}\Theta_a$.

3.2 $\mathcal{H}_2$ optimal controller and sensing design for a scaled uncertain system

In this section, we consider the $\mathcal{H}_2$ robust performance problem to guarantee a bound on the energy of the output z in response to the worst-case impulsive disturbance $w = w_i \delta(t - \tau)$, for all uncertainties $\Delta \in \Delta_a$. Let a robust performance bound $J_{\mathcal{H}_2} > 0$ be given. This is the impulse response interpretation of the $\mathcal{H}_2$ norm which as shown in [6] is equivalent to the stochastic interpretation. The $\mathcal{H}_2$ norm is then,

$$J_{\mathcal{H}_2} = \sup_{w_i, \Delta} \{ \hat{J}_{\mathcal{H}_2}(w_i, \Delta) : ||w_i|| \leq 1, \Delta \in \Delta_a \}. \tag{3.19}$$

For the worst case disturbance, we usually consider e_i to be the unit eigen vector along the i^{th} input channel. The applied input to the system is $\delta^0 e_i$, thus $x_0 = B_w e_i$. It is worth noting here that applying the weighted impulse signal $w = w_i \delta(t)$ for zero initial conditions $x(0) = 0$ is equivalent to applying an initial condition $x(0) = B_w w_i$ and zero disturbance $w(t) = 0,\ t \geq 0$. This is supported by the impulse response interpretation of the $\mathcal{H}_2$ norm as was described in (2.5).

Worst case w_i intensity is then given by the eigen vectors of $B_w^* Y B_w$ corresponding to the maximum eigenvalue. However, in our problem, the white noise does not have unit intensity and thus we consider here, $w_i e_i$, which is the weighted impulse of constant power spectral density. We assume that this intensity is constant but unknown, and it is not the worst possible disturbance but an optimally weighted one.

Looking at how to describe the upper bound $J_{\mathcal{H}_2}$ for this system $\bar{S}(P, \Delta)$,

$$||\bar{S}(P, \Delta_a^c)||_2^2 := \lim_{\tau \to \infty} \sup \frac{1}{2\tau} \int_{-\tau}^{\tau} \left(\sum_{i=1}^{n_z} ||z_i(\tau)||^2 \right) d\tau. \tag{3.20}$$

The upper bound is computed from the IQC,

$$\sup_{q \in \mathcal{L}_2} \int_0^{\infty} \left(x^T(t)(C_p^T \Theta C_p + C_y^T C_y) x(t) - q^T(t) \Theta q(t) \right) dt. \tag{3.21}$$

This optimization problem is solved using the following Riccati equation,

$$A^T Z + Z A + C_p^T \Theta C_p + C_y^T C_y + Z B_q \Theta^{-1} B_q^T Z = 0, \tag{3.22}$$

where Z is the stabilizing solution and the optimal value is

$$J_{\mathcal{H}_2} = \inf_{Z,\Theta} \mathbf{Tr}(W B_w^T Y B_w), \tag{3.23}$$

and the optimal noise spectral density is given by,

$$W = \begin{bmatrix} W_d & 0_{n_d \times n_u} & 0_{n_d \times n_y} \\ \bullet & W_a & 0_{n_u \times n_y} \\ \bullet & \bullet & 0_{n_y \times n_y} \end{bmatrix}.$$

Theorem 3. *The condition*

$$\sup_{\Delta \in \Delta_a^c} ||\bar{S}(P, \Delta)||_2^2 \leq J_{\mathcal{H}_2},$$

is equivalent to the following LMIs

$$\begin{bmatrix} A^T Z + ZA + C_p^T \Theta C_p + C_z^T C_z & ZB_q \\ B_q^T Z & -\Theta \end{bmatrix} < 0, \tag{3.24}$$

$$\begin{bmatrix} \bar{H} & (B_w \sqrt{W})^T Z \\ ZB_w \sqrt{W} & Z \end{bmatrix} > 0, \tag{3.25}$$

$$\mathbf{Tr}(\bar{H}) < \bar{J}, \tag{3.26}$$

assuming the process and actuator noise intensities are known.

Proof. Include noise intensities in the standard IQC derivation for $\|\cdot\|_2$. The $\mathcal{H}_2$ cost of the uncertain system is given as,

$$||S(P, \Delta)||_{2,imp}^2 = \limsup_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} \Big(\sum_{i=1}^{n_z} ||z_i^\tau||^2 \Big) \tag{3.27}$$

where $z_i^\tau = \bar{S}(M,\Delta)\delta^\tau w_i e_i$. The condition,

$$\sup_{\Delta\in\Delta_a^c} ||S(M,\Delta)||_{2,imp}^2 \leq J_{\mathcal{H}_2}$$

implies that

$$\sup_{\Delta\in\Delta_a^c} \sum_{i=1}^{n_z} ||z_i^\tau||^2 \leq J_{\mathcal{H}_2}$$

and to determine this upper bound, we look at the case of $\tau = 0$. The initial condition to the system is $x(0) = B_w w_i$.

Now, the optimal cost of this system is given by,

$$J(x_0) := \sup_{\Delta\in\Delta_a^c, x(0)=x_0} ||z||^2$$

such that

$$J(x_0) \leq \sup_{q\in\mathcal{L}_2[0,\infty), ||p_k||^2 \geq ||q_k||^2, x(0)=x_0} ||z||^2 \tag{3.28}$$

$$J(x_0) \leq \inf_{\theta_k>0} \sup_{q\in\mathcal{L}_2[0,\infty)} \Big(||z||^2 + \sum_{k=1}^{d} \theta_k \big(||p_k||^2 - ||q_k||^2\big)\Big) \tag{3.29}$$

Notice that the uncertainty in (3.28) is replaced by the IQC constraint in (3.29). Compute RHS of (3.29) by fixing θ_k,

$$\sup_{q\in\mathcal{L}_2[0,\infty)} \int_0^\infty \Big[x^T(t)\big(C_p^T\Theta C_p + C_z^T C_z\big)x(t) - q^T(t)\Theta q(t)\Big]dt \tag{3.30}$$

An optimal solution for this problem is obtained from the Riccati equation (3.22). If $Ric(\cdot)$

is the Riccati operator then $Z = Ric(H)$ solves (3.30) and

$$H = \begin{bmatrix} A^T & C_p^T \Theta C_p + C_z^T C_z \\ B_q \Theta^{-1} B_q^T & A \end{bmatrix},$$

then optimal cost of the integral is $x(0)^T Z x(0)$ and

$$J(x_0) \leq \inf_{Z,\Theta>0 \text{ satisfying } Ric(H)} x(0)^T Z x(0) \tag{3.31}$$

$$\sup_{\Delta \in \Delta_a^c} \sum_{i=1}^{m} ||z_i^0||^2 \leq \sum_{i=1}^{m} J(B_w w_i e_i) \tag{3.32}$$

$$= \sum_{i=1}^{n_z} \inf_{Z,\Theta>0} e_i^T w_i^T B_w^T Z B_w w_i e_i \tag{3.33}$$

$$= \inf_{Z,\Theta>0} \mathbf{Tr}(W^{T/2} B_w^T Z B_w W^{1/2}) \tag{3.34}$$

$$= \inf_{Z,\Theta>0} \mathbf{Tr}(W B_w^T Z B_w) \tag{3.35}$$

$$= J_{\mathcal{H}_2} \tag{3.36}$$

where $W_{ii} = w_i^T w_i$ is diagonal, similar to (2.6). Also note that here since the system is open loop, there is no sensor noise involved. The measurement y contains the noise w_s and it enters the system through the control input u, when the dynamic controller feedback loop is closed. □

3.2.1 Robust $\mathcal{L}_\infty$ performance

As we saw in (2.22), the system gain associated with $||z||_{\mathcal{L}_\infty}$ is related to the $\mathcal{H}_2$ performance of the system. The robust $\mathcal{L}_\infty$ performance measure is defined as the worst case value of the system in response to a bounded disturbance,

$$J_{\mathcal{L}_\infty}(\Delta, w) := \sup_{w,\Delta} \{||z||^2_{\mathcal{L}_\infty} : ||w||_{\mathcal{L}_2 \leq 1}\}. \tag{3.37}$$

3.2.2 Optimal information architecture

LMIs in (3.24), (3.25), (3.26) give a positive scaling Θ that ensures that the scaled norm of the open loop system is lesser than one.

$$\left\| \begin{bmatrix} \Theta^{1/2} & 0 \\ 0 & I \end{bmatrix} P \begin{bmatrix} \Theta^{-1/2} \\ 0 \end{bmatrix} \right\|_{\infty} < 1. \tag{3.38}$$

Next, we apply the principles of information architecture and find a controller for this scaled system to limit output covariance and control input covariance.
We find a controller for a scaled uncertain system, such that the optimization problem in (1.12), with IQC based uncertainty can be solved using the following result.

Theorem 4. *If there exists* $\gamma_s \in \mathbb{R}^{n_y} \geq 0$, $X = X^T > 0 \in \mathbb{R}^{n_x \times n_x}$, $Y = Y^T > 0 \in \mathbb{R}^{n_x \times n_x}$, $L \in \mathbb{R}^{n_u \times n_x}$, $F \in \mathbb{R}^{n_x \times n_y}$, $Z = Z^T > 0 \in \mathbb{R}^{n_x \times n_x}$, $Q \in \mathbb{R}^{n_x \times n_x}$*, and diagonal*

$\Theta \in \mathbb{R}^{n_q \times n_q}$*, that solves the optimization problem*

$$\min_{\gamma_s, X, Y, L, F, Q, Z, \Theta} \|p_s \circ \gamma_s\|_1 \tag{3.39}$$

subject to

$$\begin{bmatrix} \bar{Z} & C_z X & C_z \\ \bullet & X & I_{n_x} \\ \bullet & \bullet & Y \end{bmatrix} > 0, \tag{3.40}$$

$$\begin{bmatrix} \bar{U} & L & 0_{n_u \times n_x} \\ \bullet & X & I_{n_x} \\ \bullet & \bullet & Y \end{bmatrix} > 0, \tag{3.41}$$

$$\begin{bmatrix} \Phi_{11}^T + \Phi_{11} & \Phi_{12} \\ \bullet & \Phi_{22} \end{bmatrix} < 0, \tag{3.42}$$

and LMIs in (3.24), (3.25), (3.26)*, where*

$$\Phi_{11} := \begin{bmatrix} AX + B_u L & A \\ Q & YA + FC_y \end{bmatrix}, \tag{3.43}$$

$$\Phi_{12} := \begin{bmatrix} B_q & D_d & D_a & 0_{n_x \times n_y} \\ YB_q & YD_d & YD_a & FD_{yw} \end{bmatrix}, \tag{3.44}$$

$$\Phi_{22} := \begin{bmatrix} -\Theta & 0_{n_q \times n_d} & 0_{n_q \times n_u} & 0_{n_q \times n_y} \\ \bullet & -W_d^{-1} & 0_{n_d \times n_u} & 0_{n_d \times n_y} \\ \bullet & \bullet & -W_a^{-1} & 0_{n_u \times n_y} \\ \bullet & \bullet & \bullet & -\mathbf{diag}(\gamma_s) \end{bmatrix}, \tag{3.45}$$

assuming the process and actuator noise intensities are known, the optimal sensing archi-

tecture is defined by decision variables γ_s and the feedback controller that guarantees the $\mathcal{H}_2$ performance is recovered from

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} = \begin{bmatrix} V^{-1} & -V^{-1}YB \\ 0 & I \end{bmatrix} \begin{bmatrix} Q - YAX & F \\ L & 0 \end{bmatrix} \begin{bmatrix} U^{-1} & 0 \\ -MXU^{-1} & I \end{bmatrix}, \quad (3.46)$$

where V and U are nonsingular square matrices satisfying $YX + VU + I$.

Proof. The proof follows that in Appendix A of [1], with the scaled open-loop system

$$\begin{bmatrix} \Theta^{1/2} & 0 \\ 0 & I \end{bmatrix} P \begin{bmatrix} \Theta^{-1/2} \\ 0 \end{bmatrix},$$

where the system P is defined by (1.9) and the strictly proper dynamic output feedback controller is,

$$\dot{x}_c = A_c x_c + B_c y, \quad (3.47)$$

$$u = C_c x_c, \quad (3.48)$$

where $x_c \in \mathbb{R}^{n_c}$ is the state vector.

The closed loop system with the controller is given as,

$$\begin{bmatrix} \dot{x} \\ \dot{x_c} \end{bmatrix} = \begin{bmatrix} A + B_u D_c C_y & B_u C_c \\ B_c C_y & A_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} B_q \Theta^{-1/2} & B_w + B_u D_c D_{yw} \\ 0 & B_c D_{yw} \end{bmatrix} \begin{bmatrix} q \\ w \end{bmatrix}, \tag{3.49}$$

$$\begin{bmatrix} p \\ z \\ u \end{bmatrix} = \begin{bmatrix} \Theta^{1/2} C_p & 0 \\ C_z & 0 \\ D_c C_y & C_c \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & D_c D_{yw} \end{bmatrix} \begin{bmatrix} q \\ w \end{bmatrix}. \tag{3.50}$$

Thus, with a strictly proper controller $D_c = 0$,

$$\dot{\bar{x}} = A_{cl}\bar{x} + B_{cl}\bar{w}, \tag{3.51}$$

$$\begin{bmatrix} p \\ z \end{bmatrix} = C_{cl}\bar{x}, \tag{3.52}$$

$$u = E_{cl}\bar{x} + F_{cl}\bar{w}, \tag{3.53}$$

where $\bar{w} = \begin{bmatrix} q \\ w_d \\ w_a \\ w_s \end{bmatrix}$ and $W = \begin{bmatrix} W_d & 0_{n_d \times n_u} & 0_{n_d \times n_y} \\ \bullet & W_a & 0_{n_u \times n_y} \\ \bullet & \bullet & W_s \end{bmatrix}$.

Thus, we can write the following LMIs for information architecture.

$$A_{cl}X + XA_{cl}^T + B_{cl}WB_{cl}^T < 0, \tag{3.54}$$

$$C_{cl}XC_{cl}^T < \bar{Z}, \tag{3.55}$$

$$E_{cl}^T X E_{cl} < \bar{U}. \tag{3.56}$$

Note that in (3.55), we encounter both certain and uncertain outputs. We use (3.26) to define the upper bound on the uncertain outputs. Thus, when defining the performance measure for the controller we consider only the covariance of the certain output and (3.55) reduces to,

$$C_z X C_z^T < \bar{Z}. \tag{3.57}$$

Rest of the proof follows directly from that in Appendix A of [1]. Refer to Ch. 6 [5] to look closely at how the full order dynamic output feedback controller is reconstructed for the uncertain system.

The state equation in (3.49) is of the order of $n_c + n_x$. To reconstruct a full order ($n_c = n_x$) controller, in (3.54), (3.55) and (3.56) matrix X is assumed to be of the following structure only, and V is an arbitrary matrix

$$X = \begin{bmatrix} X_o & V \\ V & V \end{bmatrix}, \tag{3.58}$$

since if X satisfies (3.54), (3.55) and (3.56) and X_o is the order of the plant, then there always exists a transformation T_c for which the controller realization $T_c^{-1} A_c T_c, T_c^{-1}, C_c T_c$ solves (3.54), (3.55) and (3.56) for some $\tilde{X}$ that has the same structure.

Consider the transformation for the closed loop states coordinates,

$$T := \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}. \tag{3.59}$$

The new resulting states after the transformation consist of plant state x and the difference $x - x_c$. The dynamic controller estimates the plant state and thus, the difference may be thought of as estimation error. Note that when the transformation is applied,

$$\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} T^T & 0 \\ 0 & I \end{bmatrix}, \tag{3.60}$$

then if we introduce $Y_o := (X_o - V)^{-1}$ the we get the following,

$$TA_{cl}XT^T = \begin{bmatrix} AX_o + B_uC_cV & AY_o^{-1}, \\ (A - B_cC_y)X_o + (B_cC_c - A_c)V & (A - B_cC_y)Y_o^{-1} \end{bmatrix}, \tag{3.61}$$

$$TB_{cl} = \begin{bmatrix} B_q\Theta^{-1/2} & B_w \\ B_q\Theta^{-1/2} & B_w - B_cD_{yw} \end{bmatrix}, \tag{3.62}$$

$$C_{cl}XT^T = \begin{bmatrix} C_zX_o & C_zY_o^{-1} \end{bmatrix}. \tag{3.63}$$

Congruent transformation on the (3.54) yields,

$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} := T(A_{cl}X + XA_{cl}^T + B_{cl}WB_{cl}^T)T^T < 0, \tag{3.64}$$

where if we also include noise intensities,

$$\phi_{11} = AX_o + X_oA^T + B_uC_cV + VC_c^TB_u^T \tag{3.65}$$

$$+ B_q\Theta^{-1}B_q^T + B_wWB_w^T, \tag{3.66}$$

$$\phi_{12}^T = Y_o^{-1}A^T + (A - B_cC_y)X_o + (B_cC_c - A_c)V \tag{3.67}$$

$$+ (B_w - B_cD_{yw})WB_w^T, \tag{3.68}$$

$$\phi_{22} = (A - B_cC_y)Y_o^{-1} + Y_o^{-1}(A - B_cC_y)^T \tag{3.69}$$

$$+ B_q\Theta^{-1}B_q^T + (B_w - B_cD_{yw})W(B_w - B_cD_{yw})^T. \tag{3.70}$$

To find (A_c, B_c, C_c) we exploit the fact that $\phi < 0$ implies $\phi_{11} < 0$ and $\phi_{22} < 0$. Notice that these conditions are used to write (3.42),(3.43),(3.44) and (3.45). Solving these will give us the following controller,

$$A_c = A + B_u C_c - B_c C_y - Y_o^{-1}\Omega(I - X_o Y_o)^{-1}, \tag{3.71}$$

$$B_c = Y_o^{-1} C_y^T, \tag{3.72}$$

$$C_c = -B_u^T X_o, \tag{3.73}$$

where $\phi_{22} < 0$ will hold for B_c if and only if $\Omega < 0$,

$$\Omega := Y_o A + A^T Y_o + Y_o(B_w - B_c D_{yw})W(B_w - B_c D_{yw})^T Y_o + Y_o B_q \Theta^{-1} B_q^T Y_o \tag{3.74}$$

Similarly, congruent transformation on (3.55) and (3.56) will be of the form,

$$\begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} \bar{Z} & C_{cl}X \\ \bullet & X \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & T^T \end{bmatrix}, \tag{3.75}$$

$$\begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} \bar{Z} & E_{cl}X \\ \bullet & X \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & T^T \end{bmatrix}. \tag{3.76}$$

will yield (3.40) and (3.41). □

3.3 Example

Consider the convex optimization problem (3.39), subject to (3.40), (3.41), (3.42), (3.24), (3.25) and (3.26) and assume that actuator architecture is fixed, it can be solved for the tensegrity model given in (1.13).

$$Mass = 10\text{kg}, \tag{3.77}$$

$$W_d = 0.01, \tag{3.78}$$

$$Uncertainty = 10\%, \tag{3.79}$$

$$\rho_s = ones(n_y), \tag{3.80}$$

$$\bar{J} = 550. \tag{3.81}$$

The Fig.(3.1) shows that the required precision will be greater if we need a better performance. This method does not select a particularly sparse subset of sensors. However, it can be seen that the number of sensors is increasing as the $\mathcal{H}_2$ performance demanded decreases (is finer). When $||G_{w \mapsto z}|| < 10$, Fig.(3.2) shows the two sensors that need the highest precision which are the position sensors for point 6. As the degree of uncertainty is increased, we need more precise sensors.

3.3.1 Comparison to polytopic uncertainty framework

The details of this method are available in Appendix B. We compare the results of this experiment to the polytopic uncertainty method. The Fig.(3.3) shows the required sensor precisions are greater in the polytopic framework than in the IQC framework as shown in Fig.(3.2). The system in (1.13), is a Hurwitz system and thus, this method can be used to design the optimal sensing architecture for this system. It will fail for a non - Hurwitz system as the Riccati inequality (3.22) will never be satisfied.

This formulation allows us to ensure that in presence of uncertainty, the $\mathcal{H}_2$ norm of the system in response to the uncertain input does not blow up. But this can happen due to the disturbances as well. The next section decides what is the optimal precision needed to maintain closed loop disturbance rejection within a budget. Furthermore, there is one

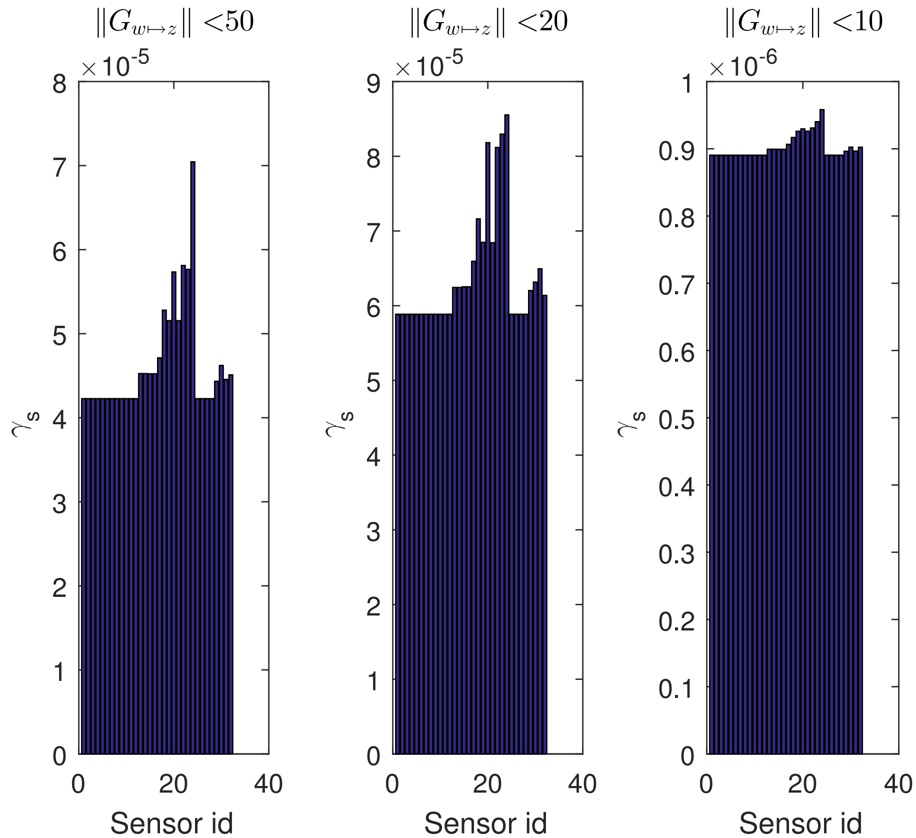

Figure 3.1: Sensing architecture for uncertain tensegrity model (1.13)

major drawback of this formulation. It requires the plant to be Hurwitz stable and will thus fail to produce a controller for an unstable uncertain system. To overcome this problem, a new kind of formulation can be proposed wherein a stabilizing controller is found for the nominal system and then a positive scaling is calculated for the uncertain controlled system such that performance criteria are met. The problem will find an optimal sensing architecture as the controller and the scaling are found from a plant which has sensing precisions as decision variables. Thus, this formulation will also be a co-design.

3.4 $\mathcal{H}_2$ optimal sensing architecture and scaling for a controlled system

The major drawback of the previous formulation was that it only worked if A is Hurwitz, thus we find a stabilizing controller C before finding the appropriate scaling for the

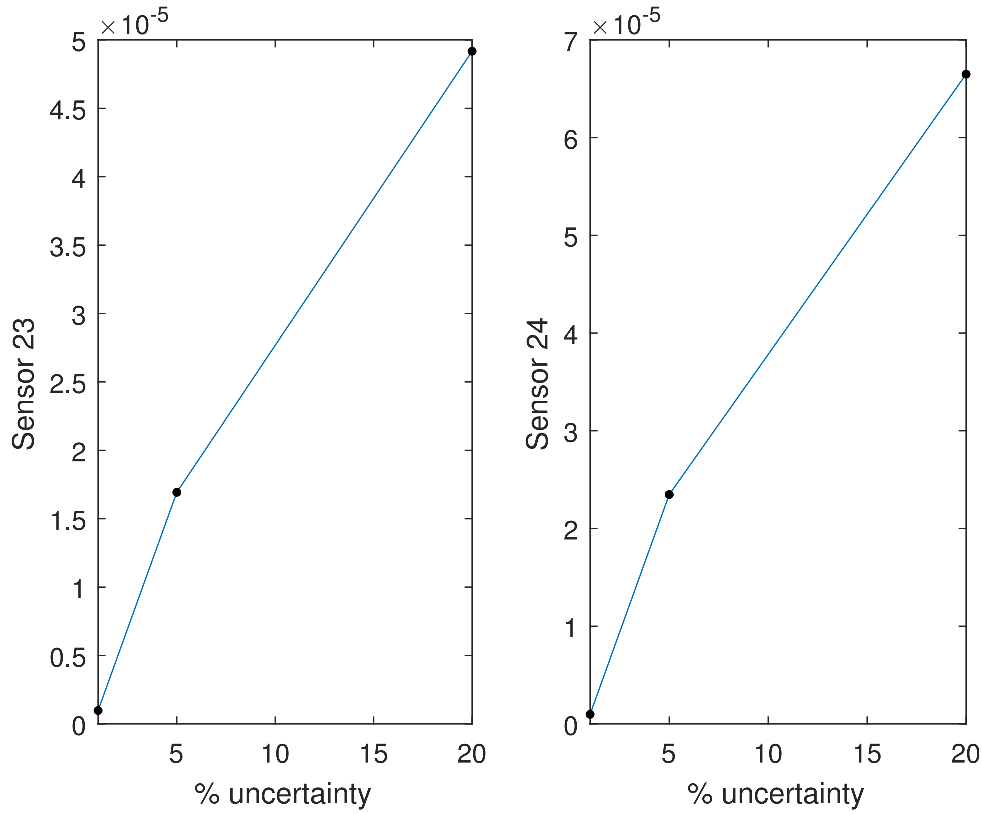

Figure 3.2: Optimal precisions with IQC Uncertainty

uncertain system. If the solution of the Riccati equation of the closed loop system exists, then it can be said that

$$\left|\left| \begin{bmatrix} \Theta^{1/2} & 0 \\ 0 & I \end{bmatrix} \underline{S}(P,C) \begin{bmatrix} \Theta^{-1/2} \\ 0 \end{bmatrix} \right|\right|_\infty < 1, \tag{3.82}$$

This implies that we guarantee robust Q-stability at the same time assuring the closed loop $\mathcal{H}_2$ performance.

3.4.1 Optimal information architecture for robust $\mathcal{H}_2$ performance

Theorem 5. *The condition*

$$\sup_{\Delta \in \Delta_a^c} ||\bar{S}(P,\Delta)||_2^2 \leq J_{\mathcal{H}_2},$$

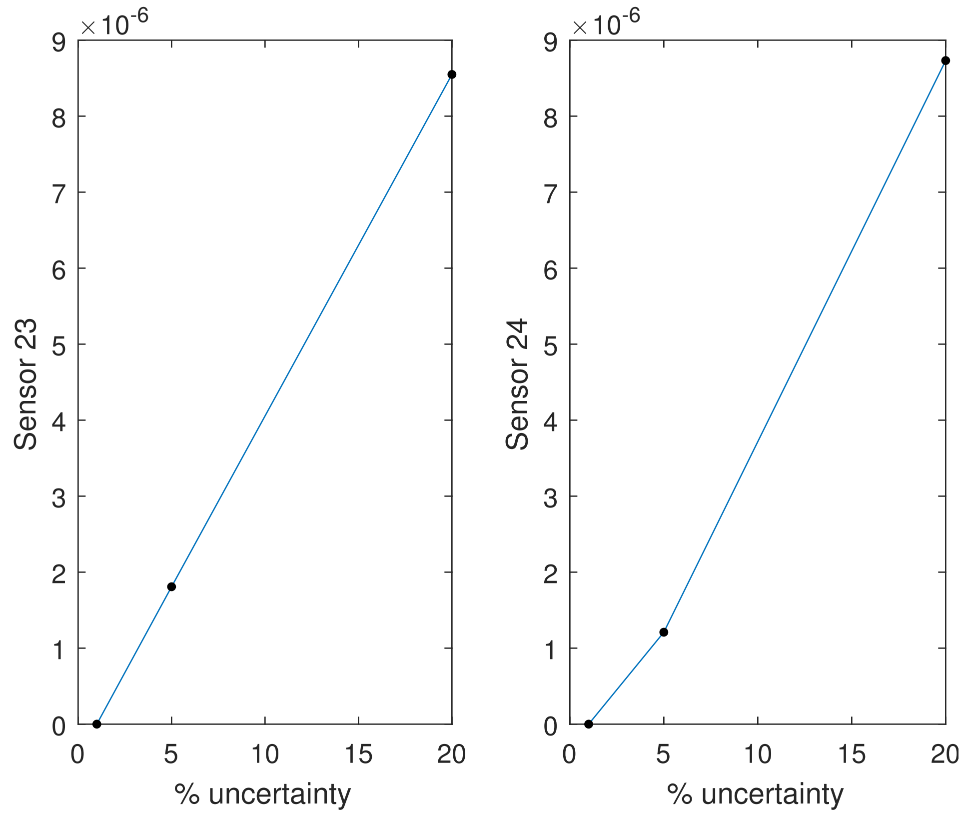

Figure 3.3: Optimal precisions with polytopic uncertainty

is equivalent to the following LMIs

$$\begin{bmatrix} A^TY + YA + FC_y + C_y^TF^T + C_p^T\Theta C_p + C_z^TC_z & YB_q \\ \bullet & -\Theta \end{bmatrix} < 0, \tag{3.83}$$

$$\mathbf{Tr}[W \begin{bmatrix} B_w^T \\ (FD_yw)^T \end{bmatrix} Y \begin{bmatrix} B_w & FD_yw \end{bmatrix}] < J_{\mathcal{H}_2}, \tag{3.84}$$

assuming the process and actuator noise intensities are known.

Proof. For a controller, $G = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}$, from Riccati equation in (3.22) and the cost equation (3.23),

$$\begin{bmatrix} A_{cl}^T Y + Y A_{cl} + C_p^T \Theta C_p + C_{cl}^T C_{cl} & Y B_q \\ B_q^T Z & -\Theta \end{bmatrix} < 0 \tag{3.85}$$

$$\mathbf{Tr}(W B_{cl}^T Y B_{cl}) < 1 \tag{3.86}$$

where $A_{cl} = A + B_u G C_y$, $B_{cl} = B_w + B_u G D_{yw}$ and $C_{cl} = C_z$. Also, the matrices B_q and C_p will be redefined as,

$$B_q = \begin{bmatrix} B_q \\ 0_{n_c \times n_q} \end{bmatrix}, \tag{3.87}$$

$$C_p = \begin{bmatrix} C_p & 0_{n_p \times n_c} \end{bmatrix}. \tag{3.88}$$

Note that the scaling is applied on the closed loop system here, and the plant P has an internally stabilizing controller. So this theorem is true for plants $P = \underline{S}(P, C)$ and the $\mathcal{H}_2$ norm can actually be written as,

$$\sup_{\Delta \in \Delta_a^c} ||\bar{S}(\underline{S}(P, C), \Delta)||_2^2 \leq J_{\mathcal{H}_2}. \tag{3.89}$$

□

To reconstruct a full-order controller, matrix Y is assumed to be of the following structure only, and V is an arbitrary matrix,

$$Y = \begin{bmatrix} Y_o & V \\ V & V \end{bmatrix}, \tag{3.90}$$

since if Y satisfies (3.83) and (3.84) and Y_o is the order of the plant, then there always exists a transformation T_c for which the controller realization $T_c^{-1}A_cT_c, T_c^{-1}, C_cT_c$ solves (3.83) and (3.84) for some $\tilde{Y}$ that has the same structure.

Consider the transformation for the closed loop states coordinates,

$$T := \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}. \tag{3.91}$$

The new resulting states after the transformation consist of plant state x and the difference $x - x_c$. The dynamic controller estimates the plant state and thus, the difference may be thought of as estimation error. Note that when the transformation is applied,

$$\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{cl}^T & C_{cl}^T \\ B_{cl}^T & D_{cl}^T \end{bmatrix} \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} T^T & 0 \\ 0 & I \end{bmatrix}, \tag{3.92}$$

then if we introduce $X_o := (Y_o - V)^{-1}$ the we get the following,

$$TA_{cl}^TYT^T =$$
$$\left[\begin{array}{c|c} A^TY_o + (B_uD_cC_y)^TY_o + (B_cC_y)^TV & A^TX_o^{-1} + (B_uD_cC_y)^TX_o^{-1} \\ \hline [A^T + (B_uD_cC_y)^T - (B_uC_c)^T]Y_o & [A^T - (B_uC_c)^T]X_o^{-1} \\ +[(B_cC_y^T - A_c^T)]V & \end{array}\right], \tag{3.93}$$

$$B_{cl}YT^T =$$
$$\left[\begin{array}{c|c} B_w^TY_o + (B_uD_cD_{yw})^TY_o + (B_cD_{yw})^TV & B_w^TX_o^{-1} + (B_uD_cD_{yw})^TX_o^{-1} \end{array}\right], \tag{3.94}$$

$$TC_{cl}^T = \left[\begin{array}{c} C_z^T \\ \hline C_z^T \end{array}\right]. \tag{3.95}$$

also here, we require,

$$B_q Y T^T = \left[\begin{array}{c|c} B_q^T Y_o & B_q^T X_o^{-1} \end{array}\right], \tag{3.96}$$

$$TC_p^T = \left[\begin{array}{c} C_p^T \\ \hline C_p^T \end{array}\right]. \tag{3.97}$$

Congruent transformation on the (3.83) yields,

$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} := T(A_{cl}^T Y + Y A_{cl} + C_{cl}^T C_{cl} + C_p^T \Theta C_p + Y B_q \Theta^{-1} B_q^T Y) T^T < 0, \tag{3.98}$$

where

$$\phi_{11} = A^T Y_o + Y_o A + V B_c C_y + C_y^T B_c^T V + Y_o B_q \Theta^{-1} B_q^T Y_o + C_z^T C_z + C_p^T \Theta C_p, \tag{3.99}$$

$$\begin{aligned} \phi_{12}^T = {} & X_o^{-1} A + A^T X_o^{-1} + (B_c C_y)^T V - (B_u C_c)^T Y_o - A_c^T V + X_o^{-1} B_q \Theta^{-1} B_q^T Y_o \\ & + C_z^T C_z + C_p^T \Theta C_p, \end{aligned} \tag{3.100}$$

$$\begin{aligned} \phi_{22} = {} & A^T X_o^{-1} + X_o^{-1} A - (B_u C_c)^T X_o^{-1} - X_o^{-1} B_u C_c + X_o^{-1} B_q \Theta^{-1} B_q^T X_o^{-1} \\ & + C_z^T C_z + C_p^T \Theta C_p. \end{aligned} \tag{3.101}$$

To find (A_c, B_c, C_c) we exploit the fact that $\phi < 0$ implies $\phi_{11} < 0$ and $\phi_{22} < 0$. Using congruent transformation, we write $\phi_{22} < 0$ as $X_o \phi_{22} X_o < 0$.

$$A^T Y_o + Y_o A + V B_c C_y + C_y^T B_c^T V + Y_o B_q \Theta^{-1} B_q^T Y_o + C_z^T C_z + C_p^T \Theta C_p < 0, \tag{3.102}$$

$$A X_o + X_o A^T - X_o (B_u C_c)^T - B_u C_c X_o + B_q \Theta^{-1} B_q^T + X_o C_z^T C_z X_o + X_o C_p^T \Theta C_p X_o < 0. \tag{3.103}$$

Express these as LMIs,

$$\begin{bmatrix} A^T Y_o + Y_o A + V B_c C_y + C_y^T B_c^T V + C_z^T C_z + C_p^T \Theta C_p & Y_o B_q \\ \bullet & \Theta \end{bmatrix} < 0, \tag{3.104}$$

$$\begin{bmatrix} A X_o + X_o A^T - X_o (B_u C_c)^T - B_u C_c X_o + B_q \Theta^{-1} B_q^T & X_o C_p^T & X_o C_z^T \\ \bullet & -\Theta^{-1} & 0_{n_z \times n_q} \\ \bullet & \bullet & -I_{n_z \times n_z} \end{bmatrix} < 0. \tag{3.105}$$

A closer look at these conditions will reveal that these inequalities are non-convex. To find a dynamic controller, we need to simultaneously solve these two equations. They contain both Θ and its inverse Θ^{-1}. There is also a pair of variables that tend to inverse of each other Y_o and X_o. There is no known mathematical trick that can turn these equations into a convex form. They are also not linear. The only way to solve these is to treat them as Bi-linear Matrix Inequalities (BMI) and use other methods to reach an optimal solution. This means that there is no guarantee of a global minimum and that we need to employ an iterative search till we arrive at the minimal solution to this problem. This is not as attractive a method as LMIs.

One other way to treat this problem is to assume that we know the controller and use the set of equations (3.83) and (3.84) to find an observer $G = Y^{-1}F$ such that the problem is solved. The next section works along a similar line of thought. It divides the optimization into two steps and finds one dynamic output feedback controller.

3.4.2 A two step convex optimization algorithm

Step One: Solve the noiseless state feedback problem for uncertain system to determine minimum possible output covariance, control covariance and positive scaling.

Theorem 6. *If there exists $X = X^T > 0 \in \mathbb{R}^{n_x \times n_x}$, $L \in \mathbb{R}^{n_u \times n_x}$, and diagonal $\Theta \in \mathbb{R}^{n_q \times n_q}$, that solves the optimization problem*

$$\begin{bmatrix} AX + XA^T + B_uL + L^TB_u^T + B_wWB_w^T + B_q\Theta B_q^T & XC_p^T \\ \bullet & -\Theta \end{bmatrix} < 0, \tag{3.106}$$

$$\begin{bmatrix} \bar{Z} & C_zX \\ \bullet & X \end{bmatrix} > 0, \tag{3.107}$$

$$\begin{bmatrix} \bar{U} & L \\ \bullet & X \end{bmatrix} > 0, \tag{3.108}$$

we can say the system is robustly stable and the best possible $\mathcal{H}_2$ norm is $J_{\mathcal{H}_2} = \mathbf{Tr}\bar{Z} + \mathbf{Tr}\bar{U}$, assuming the process and actuator noise intensities are known.

Proof. Consider the closed loop system, where the controller is state feedback type and there is no noise $W_s = 0$,

$$\dot{\bar{x}} = A\bar{x} + B_uu, \tag{3.109}$$

$$z = C_z\bar{x}, \tag{3.110}$$

$$u = K\bar{x}. \tag{3.111}$$

We need to write the information architecture problem for static gain feedback for this uncertain system. From Theorem 4.7.3 in [5], which gives the condition for Q-stability based on a controllability gramian-like Riccati equation.

$$A_{cl}X + XA_{cl}^T + B_wWB_w^T + B_q\Theta B_q^T + XC_p^T\Theta^{-1}C_pX < 0, \tag{3.112}$$

$$C_{cl}^T X C_{cl} < \mathbf{Tr}\bar{Z}, \tag{3.113}$$

$$E_{cl}^T X E_{cl} < \mathbf{Tr}\bar{U}. \tag{3.114}$$

where $A_{cl} = A + B_uK$, $B_{cl} = B_w$, $C_{cl} = C_z$ and $E_{cl} = K$.

The goal in this problem to find a scaling such that the $J_{\mathcal{H}_2}$ performance is achieved. The problem originally solves for robust $\mathcal{L}_\infty$ performance (3.37) and thus, for finite-energy disturbances [5]. We know from (2.22) that the $\mathcal{L}_\infty$ norm of output relates to the $\mathcal{H}_2$ norm of the system. If we impose a causality restriction on the uncertainty set, the $\mathcal{L}_\infty$ measure for the known information architecture can be used to define the upper bound on the $\mathcal{H}_2$ performance on the optimal information architecture problem. (In the next step, we assume the noise intensity is unknown.)

To simplify the problem, we have assumed that the actuator architecture is known. Thus, in (3.112), since $W_s = 0$ and W_d and W_a are specified, then W is known. So, this problem is convex in X and Θ. The LMI follows from (3.105). □

Step Two: Assuming the scaling is given, find a stabilizing dynamic controller and optimal sensor precision such that the performance of the system is bound by that found in the previous step.

Theorem 7. *If there exists* $\gamma_s \in \mathbb{R}^{n_y} \geq 0$, $X = X^T > 0 \in \mathbb{R}^{n_x \times n_x}$, $Y = Y^T > 0 \in \mathbb{R}^{n_x \times n_x}$, $F \in \mathbb{R}^{n_x \times n_y}$, *for a diagonal scaling* $T = \Theta^{-1} \in \mathbb{R}^{n_q \times n_q}$, *that solves the* *optimization problem*

$$\min_{\gamma_s, X, Y, F} \|p_s \circ \gamma_s\|_1 \tag{3.115}$$

subject to

$$\begin{bmatrix} XA^T + AX + B_uL + L^TB_u^T + B_q\Theta B_q^T & XC_p^T & XC_z^T \\ \bullet & -\Theta & 0_{n_z \times n_q} \\ \bullet & \bullet & -I_{n_z \times n_z} \end{bmatrix} < 0, \tag{3.116}$$

$$\begin{bmatrix} A^TY + YA + FC_y + C_y^TF^T + C_p^TTC_p + C_z^TC_z & YB_q \\ \bullet & -T \end{bmatrix} < 0, \tag{3.117}$$

$$\begin{bmatrix} -(\frac{J_{\mathcal{H}_2}}{n_w})W_d^{-1} & 0_{n_d \times n_u} & 0_{n_d \times n_y} & D_d^TY & D_d^T \\ \bullet & -(\frac{J_{\mathcal{H}_2}}{n_w})W_a^{-1} & 0_{n_u \times n_y} & D_a^TY & D_a^T \\ \bullet & \bullet & -(\frac{J_{\mathcal{H}_2}}{n_w})\textbf{diag}(\gamma_s) & D_y^TF^T & 0_{n_y \times n_x} \\ \bullet & \bullet & \bullet & Y & I_{n_x \times n_x} \\ \bullet & \bullet & \bullet & I_{n_x \times n_x} & X \end{bmatrix} > 0, \tag{3.118}$$

assuming the process and actuator noise intensities are known, the optimal sensing architecture is defined by precisions γ_s and the feedback controller that guarantees the $\mathcal{H}_2$ performance is recovered from

$$Ac = -A^TX^{-1} - YA - FC_y - YB_uLX^{-1} - YB_qT^{-1}B_q^TX^{-1} - C_p^TTC_p - C_z^TC_z \tag{3.119}$$

$$Bc = -F \tag{3.120}$$

$$Cc = -LX^{-1} \tag{3.121}$$

where the transfer function of the controller is $C_c(s(X^{-1} - Y) - A_c)B_c$.

Proof. In this problem, T and Θ are known from the previous step. This helps overcome the non-convexity issue.

The LMI in (3.116) follows from (3.112) in the last step. Similarly, the LMI in (3.117) is derived from the condition (3.104).

The equations (3.107) and (3.108) describe the closed loop $\mathcal{H}_2$ norm of the system. We know that for any system it is true that,

$$\mathbf{Tr}[C_{cl}XC_{cl}^T] = \mathbf{Tr}[WB_{cl}^TYB_{cl}] < J_{\mathcal{H}_2}. \tag{3.122}$$

Here, the sensing architecture is unknown and $W = \begin{bmatrix} W_d & 0_{n_d \times n_u} & 0_{n_d \times n_y} \\ \bullet & W_a & 0_{n_u \times n_y} \\ \bullet & \bullet & W_s \end{bmatrix}$.

Look closely at this new cost inequality, remember that W is diagonal and is positive definite $W > 0$,

$$\mathbf{Tr}[WB_{cl}^TYB_{cl}] < J_{\mathcal{H}_2}, \tag{3.123}$$

$$\mathbf{Tr}[WB_{cl}^TYB_{cl}] < J_{\mathcal{H}_2}(\frac{n_w}{n_w}), \tag{3.124}$$

$$\mathbf{Tr}[WB_{cl}^TYB_{cl}] < \frac{J_{\mathcal{H}_2}}{n_w}\mathbf{Tr}I_{n_w \times n_w}, \tag{3.125}$$

$$W^{T/2}B_{cl}^TYB_{cl}W^{1/2} < \frac{J_{\mathcal{H}_2}}{n_w}I_{n_w \times n_w}, \tag{3.126}$$

Congruent transformation by $W^{-1/2} > 0$,

$$B_{cl}^T Y B_{cl} < W^{-T/2}(\frac{J_{\mathcal{H}_2}}{n_w} I_{n_w \times n_w}) W^{-1/2}, \tag{3.127}$$

$$B_{cl}^T Y B_{cl} < \frac{J_{\mathcal{H}_2}}{n_w} W^{-1}, \tag{3.128}$$

$$\frac{J_{\mathcal{H}_2}}{n_w} W^{-1} - B_{cl}^T Y B_{cl} > 0 \tag{3.129}$$

$$\begin{bmatrix} \frac{J_{\mathcal{H}_2}}{n_w} W^{-1} & B_{cl}^T Y \\ \bullet & Y \end{bmatrix} > 0 \tag{3.130}$$

Congruent transformation by T_c on (3.130) will be of the form,

$$\begin{bmatrix} I & 0 \\ 0 & T \end{bmatrix} \begin{bmatrix} \frac{J_{\mathcal{H}_2}}{n_w} W^{-1} & B_{cl}^T Y \\ \bullet & Y \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & T^T \end{bmatrix}, \tag{3.131}$$

will yield (3.118).

To reconstruct the controller, we use the equations (3.101), (3.100) and (3.99). Recall that,

$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} := T(A_{cl}^T Y + Y A_{cl} + C_{cl}^T C_{cl} + C_p^T \Theta C_p + Y B_q \Theta^{-1} B_q^T Y) T^T < 0, \tag{3.132}$$

and using Schur's Lemma, we know that $\phi < 0$ implies that $\phi_{11} < 0$ and $\phi_{22} < 0$. We eliminate some of the controller parameters and reduce the controller design to a convex problem. So, we eliminate A_c. If $\phi_{11} < 0$ and $\phi_{22} < 0$, then there exists A_c such that $\phi < 0$. One choice is choosing $\phi_{12} = 0$ as follows,

$$Ac = -A^T X^{-1} - YA - FC_y - YB_u L X^{-1} - YB_q T^{-1} B_q^T X^{-1} - C_p^T T C_p - C_z^T C_z. \tag{3.133}$$

Next, we eliminate C_c which only appears in ϕ_{22}, by completing the square in ϕ_{22},

$$0 = (C_c - B_u X_o^{-1})^T (C_c - B_u X_o^{-1}) + \Omega, \tag{3.134}$$

$$\Omega = A^T X_o^{-1} + X_o^{-1} A + X_o^{-1} B_q \Theta^{-1} B_q^T X_o^{-1} - X_o^{-1} B_u B_u^T X_o^{-1} + C_z^T C_z + C_p^T \Theta C_p. \tag{3.135}$$

This inequality $\phi_{22} < 0$ holds for some C_c if and only if $\Omega < 0$ holds, in this situation a choice of C_c is given by,

$$C_c = B_u X_o^{-1}. \tag{3.136}$$

Inequality (3.117) follows by letting $F := VB_c$ in (3.104). Thus,

$$B_c = -F, \tag{3.137}$$

where the controller is a descriptor model,

$$(X^{-1} - Y)\dot{x}_c = A_c x_c + B_c y, \tag{3.138}$$

$$u = C_c x_c. \tag{3.139}$$

$\square$

3.5 Example

We apply the two step algorithm for co-design of information architecture and controller discussed in this section for uncertain systems. Consider the convex optimization problem (3.115), subject to (3.116), (3.117) and (3.118), provided the first step has solved the equations (3.106), (3.107) and (3.108) then we can solve the robust $\mathcal{H}_2$ problem for the tensegrity model given in (1.13). Assume that actuator architecture is fixed and the other parameters for the experiment are given as,

$$Mass = 10\text{kg}, \tag{3.140}$$

$$W_d = 0.01, \tag{3.141}$$

$$\rho_s = ones(n_y). \tag{3.142}$$

In Fig.(3.4), we fix the $Uncertainty = 20\%$, and find the optimal sensing architecture for different values of $J_{\mathcal{H}_2}$. In Fig.(3.5), we fix the $J_{\mathcal{H}_2} < 1$, and find the optimal sensing

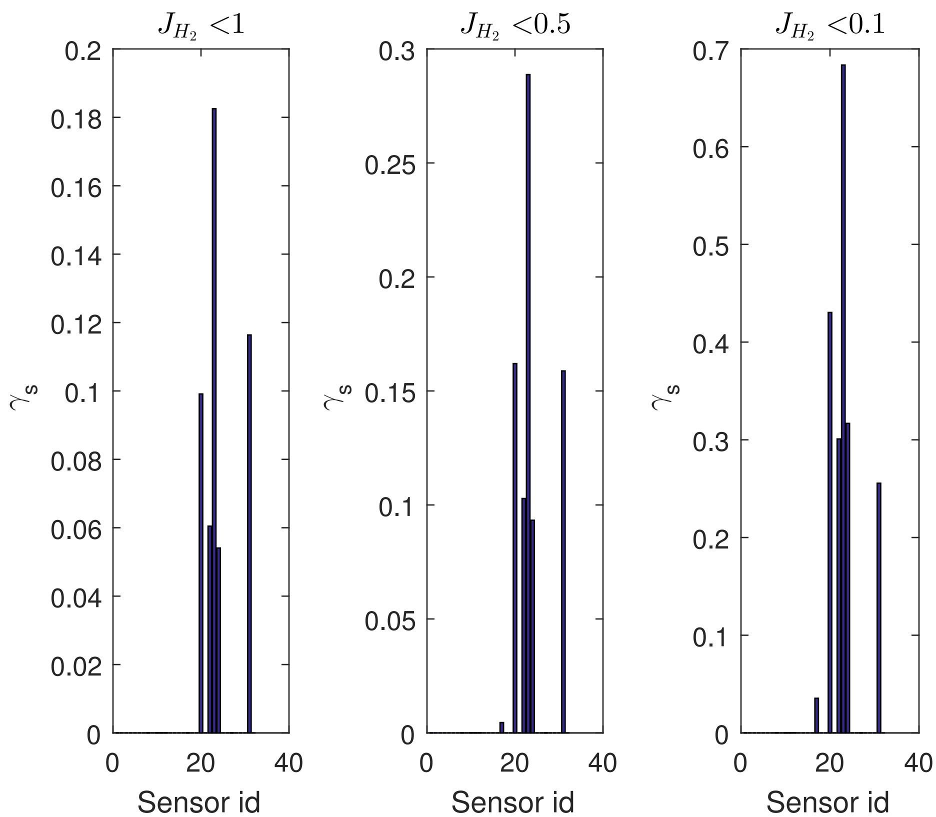

Figure 3.4: Sensing architecture for uncertain system (1.13)

precision for sensors chosen in Fig.(3.4) for different percentages of uncertainty. We notice that as we increase the uncertainty, the required sensor precision increases.

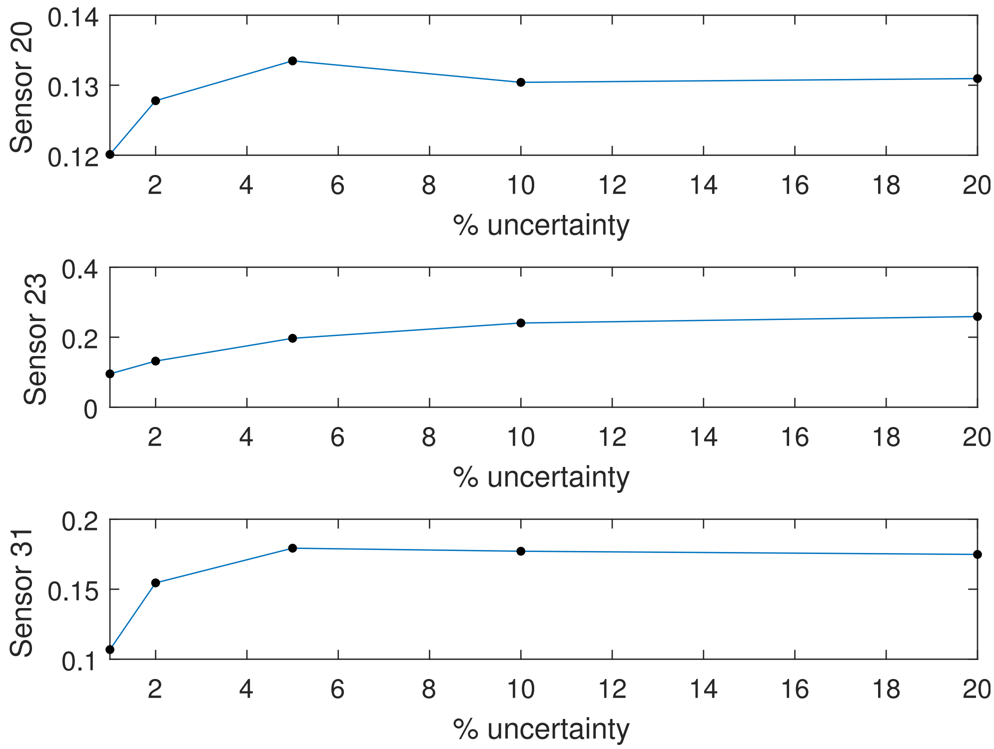

Figure 3.5: Sensor precisions for varying uncertainty (1.13)

3.5.1 Robust $\mathcal{H}_2$ performance with minimal set of sensors

If we wish to find the minimal set of sensors that will achieve the performance, we can do the following procedure:

1. Eliminate the sensor with the least required precision
2. Run the two-step algorithm. If feasible, go back to Step 1. Else if infeasible, this is the optimal set of sensors to achieve the given performance.

Assume that actuator architecture is fixed and the other parameters for the experiment are given as,

$$Mass = 10\text{kg}, \tag{3.143}$$

$$W_d = 0.01, \tag{3.144}$$

$$Uncertainty = 20\%, \tag{3.145}$$

$$\rho_s = ones(n_y), \tag{3.146}$$

$$J_{\mathcal{H}_2} = 0.1. \tag{3.147}$$

The system has only the sensors chosen by the optimization, thus,

$$SensorArray = \begin{pmatrix} Sensor20 \\ Sensor22 \\ Sensor23 \\ Sensor31 \end{pmatrix}. \tag{3.148}$$

Fig.(3.4) displays the optimal precisions of the chosen sensors.

The robust performance is met ($J_{\mathcal{H}_2} < 0.1$) as shown below,

$$||G_{w \mapsto z}|| = 0.02 \tag{3.149}$$

$$||\Theta^{1/2} P_{11} \Theta^{-1/2}|| = 0.04. \tag{3.150}$$

The norm of the scaled system tends to zero, and also additionally that the norm of the controlled uncertain system without applying the scaling is finite,

$$||\bar{S}(P, \Delta)|| = 19.28. \tag{3.151}$$

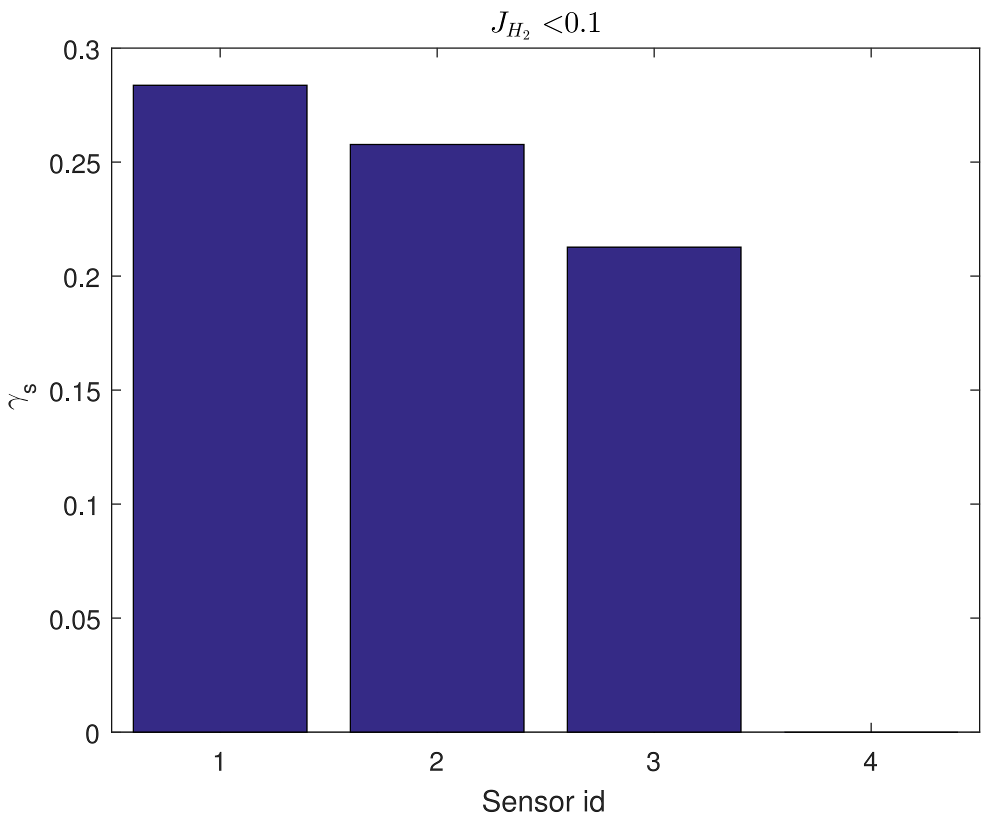

Figure 3.6: Sensing architecture for minimal system (1.13)

4. SUMMARY AND CONCLUSIONS

In this work, a framework that integrates control system design and estimation with the optimal information architecture for uncertain systems was presented. In the first method, two approaches to this problem were presented. The first approach is based on polytopic representation of the uncertainty and provides a conservative formulation. In the second approach, the uncertainty is modeled using integral quadratic constraints and provide a much better framework. We apply both the algorithms to an optimal sensing and control design of active suspension problem. In the second method, a completely new formulation was proposed to address the shortcomings of the first method. A scaling for a stabilized system assured closed loop robust $\mathcal{H}_2$ performance. The uncertainty is modelled using IQCs. Due to issues of linearity and convexity of the optimization, a new algorithm was suggested that solves the l_1 regularization problem along with the stabilizing $\mathcal{H}_2$ optimal performance problem in an convex method. We apply this new algorithm to the tensegrity model and observe that finer outer covariance calls for more precise sensors. In both the methods, the l_1 regularization selects an optimal subset of the included sensors. We also observed the required sensing precision increases with increasing model uncertainty.

4.1 Challenges

The thesis operates entirely within the linear time-invariant domain. It employs LMIs to find optimal solutions to convex problems. It is not always possible to frame every problem as a linear convex optimization. This was the biggest challenge to face. As we saw in Section 3, the presence of non-convexity means that there is no guaranteed minimal solution, which means running an iterative algorithm to see if such a point ever occurs. The challenge was overcome by splitting the optimization in two. The two step algorithm proposed a way to avoid this and divided the non-convex problems into two separate convex ones, and arrived at a minimum solution.

4.2 Further study

The design of information architecture is a very promising field. There are several ways to extend this work and make the formulation stronger.

1. Adding sparsity promoting terms for controller : This thesis proposes optimal sensing architecture for uncertain systems, however to have system level optimization even the actuation architecture needs to be optimal. l_1 regularization can be employed to induce sparsity in the communication links in a decentralized controller.

2. Separating direct feed through terms, to forgo filters for uncertain outputs : We forced certain assumptions on the system, particularly that certain D terms on the transfer function were assumed to be zero. This might be the case in all systems, during practice these assumptions might not hold and certain measures like filtering would need to be enforced. However, a possible direction is to take care of these terms during design, by separating the system into two parts, one having direct feed through and the other not and applying $\mathcal{H}_2$ minimization separately.

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APPENDIX A

LINEAR MATRIX INEQUALITIES

Linear matrix inequalities are a mathematical tool that has been used for convex optimization problems. LMIs can be solved using a number of methods, in this work we use the default solver SDPT3 in *'cvx'*. Most common control problems can be expressed in terms of LMIs, including but not limited to linear quadratic regulator problem, standard $\mathcal{H}_2$ problem and Riccati inequalities.

A linear matrix inequality has the form [12],

$$F(x) := F_0 + \sum_{i=1}^{m} x_i F_i > 0, \tag{A.1}$$

where $x \in \mathbb{R}$ is the variable and the symmetric matrices $F_i = F_i^T \in \mathbb{R}^n$ for $n, i = 0, \cdots, m$, are given. The inequality symbol in (A.1) means that $F(x)$ is positive-definite, i.e., $u^T F(x) u > 0$ for all nonzero $u \in \mathbb{R}^n$.

A.1 Schur's lemma

When the matrices F_i are diagonal, the LMI $F(x) > 0$ is equivalent to a set of linear inequalities. Schur's complement method is used to make the non-linear inequa;ities linear. The basic idea is as follows:

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0, \tag{A.2}$$

where $Q(x) = Q^T(x)$,$R(x) = R^T(x)$, and $S(x)$ depend affinely on x, is equivalent to,

$$R(x) > 0, \qquad Q(x) - S(x)R^{-1}(x)S^T(x) > 0, \tag{A.3}$$

$$Q(x) > 0, \qquad R(x) - S^T(x)Q^{-1}(x)S(x) > 0. \tag{A.4}$$

This is used to make several inequalities that are quadratic in one LMI variable, linear in the same.

APPENDIX B

INFORMATION ARCHITECTURE WITH POLYTOPIC UNCERTAINTY

Consider the system in (1.6), (1.7) and (1.8). We assume $\Delta \in \mathcal{P}_{\{\Delta_i\}}$, where $\mathcal{P}_{\{\Delta_i\}}$ is a polytope defined by vertices $\{\Delta_i\}$ and present the following result.

Theorem 8. *The optimization problem in* (1.12) *subject to* (1.11) *is equivalent to the following convex optimization problem*

$$\min_{\gamma_s, X, Y, L, F, Q} \|p_s \circ \gamma_s\|_1 \tag{B.1}$$

subject to

$$\begin{bmatrix} \bar{Z} & C_z(\Delta_i)X & C_z(\Delta_i) \\ \bullet & X & I_{n_x} \\ \bullet & \bullet & Y \end{bmatrix} > 0, \tag{B.2}$$

$$\begin{bmatrix} \bar{U} & L & 0_{n_u \times n_x} \\ \bullet & X & I_{n_x} \\ \bullet & \bullet & Y \end{bmatrix} > 0, \tag{B.3}$$

$$\begin{bmatrix} \Phi_{11}^T(\Delta_i) + \Phi_{11}(\Delta_i) & \Phi_{12}(\Delta_i) \\ \bullet & \Phi_{22}(\Delta_i) \end{bmatrix} < 0, \tag{B.4}$$

where

$$\Phi_{11}(\Delta_i) := \begin{bmatrix} A(\Delta_i)X + B_u(\Delta_i)L & A(\Delta_i) \\ Q & YA(\Delta_i) + FC_y(\Delta_i) \end{bmatrix}, \quad \text{(B.5)}$$

$$\Phi_{12}(\Delta_i) := \begin{bmatrix} D_d(\Delta_i) & D_a(\Delta_i) & 0_{n_x \times n_y} \\ YD_d(\Delta_i) & YD_a(\Delta_i) & FD_y(\Delta_i) \end{bmatrix}, \quad \text{(B.6)}$$

$$\Phi_{22}(\Delta_i) := \begin{bmatrix} -W_d^{-1} & 0_{n_d \times n_u} & 0_{n_d \times n_y} \\ \bullet & -W_a^{-1} & 0_{n_u \times n_y} \\ \bullet & \bullet & -\mathbf{diag}(\gamma_s) \end{bmatrix}, \quad \text{(B.7)}$$

for all Δ_is that define the polytopic uncertainty. The variables in the optimization are $\gamma_s \in \mathbb{R}^{n_y} \geq 0$, $X = X^T > 0 \in \mathbb{R}^{n_x \times n_x}$, $Y = Y^T > 0 \in \mathbb{R}^{n_x \times n_x}$, $L \in \mathbb{R}^{n_u \times n_x}$, $F \in \mathbb{R}^{n_x \times n_y}$, and $Q \in \mathbb{R}^{n_x \times n_x}$. The symbol $\bullet$ represents the symmetric terms.

Proof. Here, $A(\Delta_i)$ can be expressed as $A_0 + \Delta_i A$, where $\{\Delta_i\}$ are the vertices of the polytope, and similarly for all the uncertain matrices. Thus, we have i sets of matrices and we apply the information architecture LMIs from [1] to get one controller. $\square$